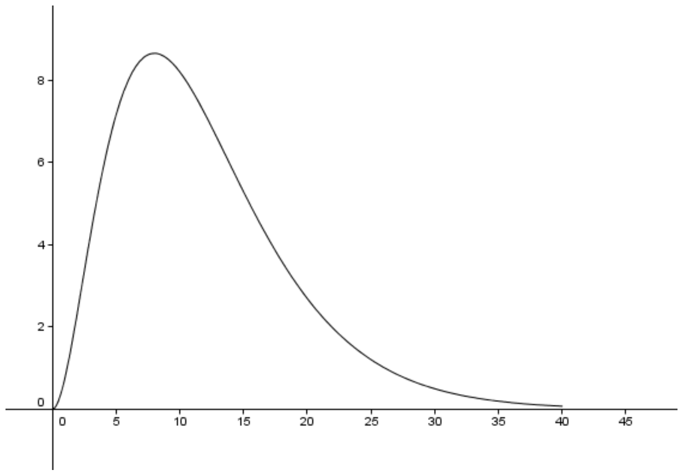
Math 1314
Lesson 12
Curve Analysis (Polynomials)

This lesson will cover analyzing polynomial functions using GeoGebra.

Suppose your company embarked on a new marketing campaign and was able to track sales based on it. The graph below gives the number of sales in thousands shown *t* days after the campaign began.



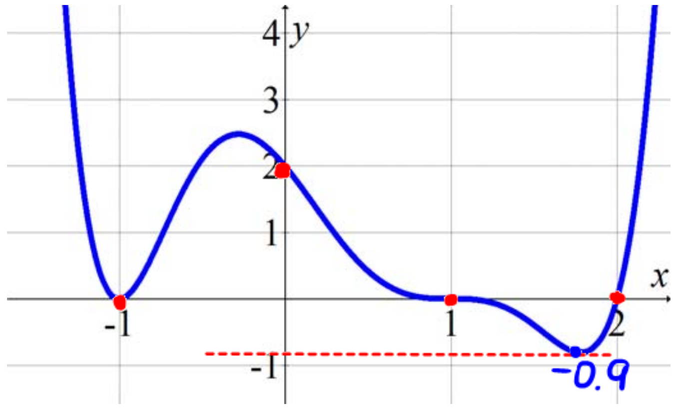
Now suppose you are assigned to analyze this information. We can use calculus to answer the following questions:

When are sales increasing or decreasing?
What is the maximum number of sales in the given time period?
Where does the growth rate change?
Etc.

Calculus can't answer the "why" questions, but it can give you some information you need to start that inquiry.

There will be several features of a polynomial function that we'll need to find and some are easier found by using calculus. Let's start with a few College Algebra topics.

Example 1: The graph below is the graph of a polynomial function. State whether each given statement is true or false.



I. The domain of the polynomial function is $(-\infty, \infty)$.

True

II. The range is $(-\infty, \infty)$. False

$[-0.9, \infty)$

III. The roots of the polynomial are -1, 1 and 2.

True

IV. The *y*-intercept is 0. False

y-int = 2

V. The function has one relative maximum and two relative minimum.

True

VI. The function is an even degree polynomial.

True

VII. The function's leading coefficient is negative.

False

Intervals on Which a Function is Increasing/Decreasing

A function is **increasing** on an interval (a, b) if, for any two numbers x_1 and x_2 in (a, b), $f(x_1) < f(x_2)$, whenever $x_1 < x_2$. A function is **decreasing** on an interval (a, b) if, for any two numbers x_1 and x_2 in (a, b), $f(x_1) > f(x_2)$, whenever $x_1 < x_2$.

In other words, if the *y* values are getting bigger as we move from left to right across the graph of the function, the function is increasing. If they are getting smaller, then the function is decreasing. We will state intervals of increase/decrease using interval notation. The interval notation will consists of corresponding *x*-values wherever *y*-values are getting bigger/smaller.

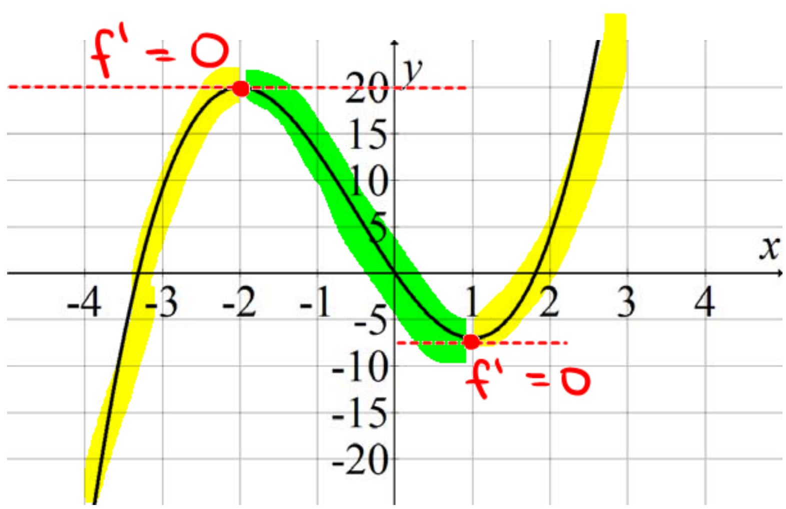
Example 2: Given the following graph of a function, state the intervals on which the function is:

a. increasing.

$(-\infty, -2) \cup (1, \infty)$

b. decreasing.

$(-2, 1)$

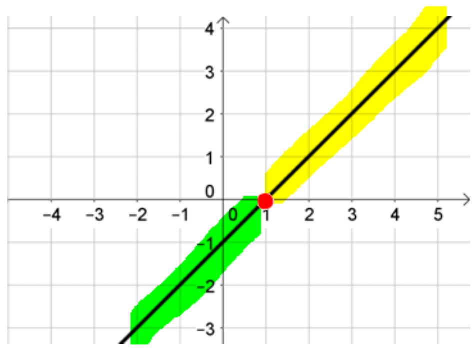


Do you notice WHERE increasing/decreasing CHANGES?

We can use calculus to determine where a function changes from increasing to decreasing or from decreasing to increasing, this occurs at its **critical numbers**. The **critical numbers** of a polynomial function are all values of x that are in the domain of f where $f'(x)=0$ (the tangent line to the curve is horizontal). A relative maximum or a relative minimum can only occur at a **critical number.**

A function is **increasing** on an interval if the **first derivative** of the function is **positive** for every number in the interval. A function is **decreasing** on an interval if the **first derivative** of the function is **negative** for every number in the interval.

Example 3: The graph given below is the *first derivative* of a function, f.



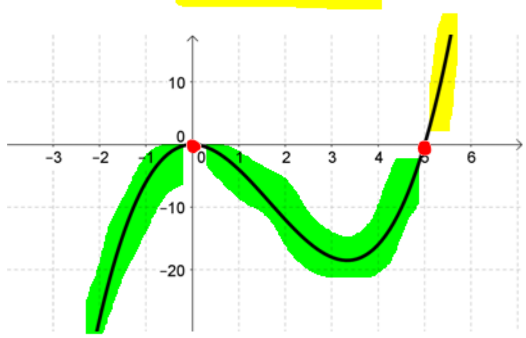
a. Find any critical numbers. $f'=0$ $x=1$

b. Find any intervals where the function f is:

increasing. $f'>0$ $(1,\infty)$

decreasing. $f'<0$ $(-\infty,1)$

Example 4: The graph given below is the *first derivative* of a function, f.



a. Find any critical numbers. $f'=0$

$x=0$ & $x=5$

b. Find any intervals where the function f is:

increasing. $f'>0$

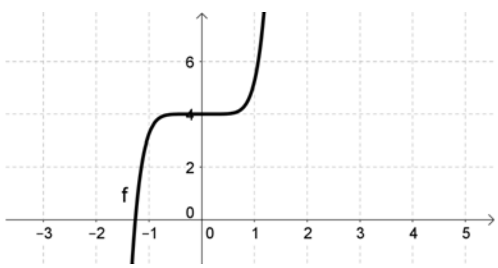
decreasing. $f'<0$

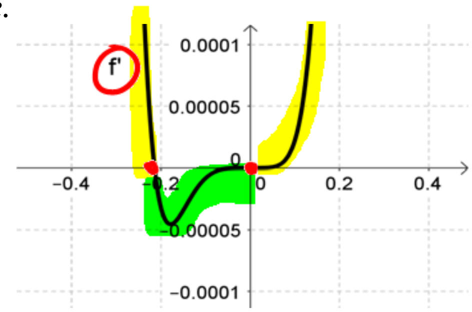
$(5,\infty)$ $(-\infty,5)$

To find the intervals on which a given polynomial function is increasing/decreasing using GGB:

1. Use GGB to graph the derivative of the function.
2. Find any critical numbers. (*Recall that the critical numbers occur whenever* $f'(x)=0$; *hence, simply find the zeros of* f'.)
3. Create a number line, subdividing the line using the critical numbers.
4. Use the graph of the derivative to determine the sign (positive or negative) of the *y* values of the derivative in each interval and record this on your number line.
5. Determined the intervals of increase/decrease.

Example 5: Let $f(x)=x^7+0.25x^6+4$. Find where the function is increasing/decreasing.
Enter the function into GGB then find its derivative.





Command: root[f'(x)] (-0.2143, 0) & (0, 0)

f: ↗ ↘ ↗

f': + − +

-0.2143 0

Increasing: $(-\infty,-0.2143)\cup(0,\infty)$

Decreasing: $(-0.2143, 0)$

Relative Extrema Using Calculus

You can use the same number line that you created to determine intervals of increase/decrease to find the *x* coordinate of any relative extrema. Decreasing to Increasing, you have a relative minimum. Increasing to Decreasing, you have a relative maximum. These may only occur at critical numbers.

Example 6: Give any extremum for Example 5.

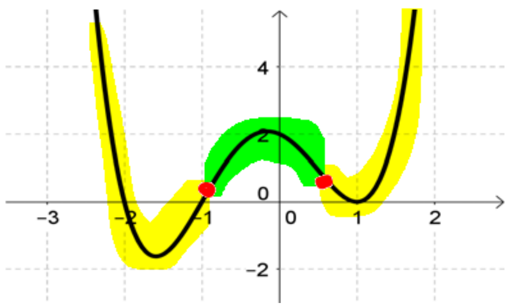
Relative Max: $(-0.2143, 4.000003)$ ↑ $f(-0.2143)$

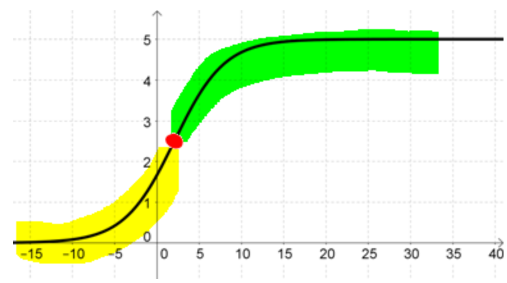
Relative Min: $(0, 4)$ ↑ $f(0)$

down up

Concavity

In business, for example, the first derivative might tell us that our sales are increasing, but the second derivative will tell us if the pace of the increase is increasing or decreasing.





From these graphs, you can see that the shape of the curve change differs depending on whether the slopes of tangent lines are increasing or decreasing. This is the idea of **concavity**.

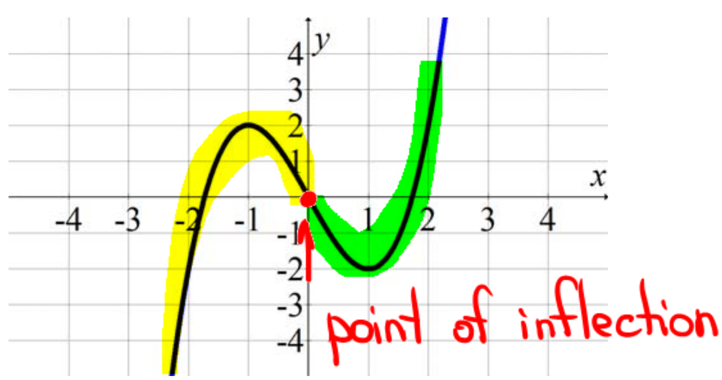
Example 7: The graph given below is the graph of a function f. Determine the interval(s) on which the function is:

a. concave upward.

$(0, \infty)$

b. concave downward.

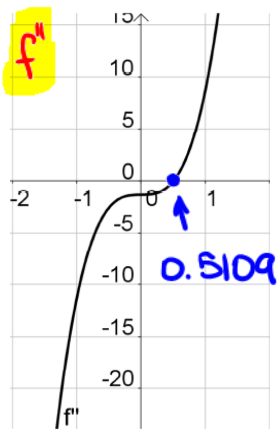
$(-\infty, 0)$



We find concavity intervals by analyzing the *second derivative* of the function. The analysis is very similar to the method we used to find increasing/decreasing intervals.

1. Use GeoGebra to graph the second derivative of the function. Then find the zero(s) of the second derivative.
2. Create a number line and subdivide it using the zeros of the second derivative.
3. Use the graph of the second derivative to determine the sign (positive or negative) of the *y* values of the second derivative in each interval and record this on your number line.
4. In each interval in which the second derivative is positive, the function is concave upward. In each interval in which the second derivative is negative, the function is concave downward.

Example 8: State intervals on which the function is concave upward and intervals on which the function is concave downward: $f(x) = \frac{1}{2}x^5 - \frac{2}{3}x^2 - 8x - 1$



Command: root [f''] $x = 0.5109$

f: ∩ ∪

f'': − +

$-\infty$ 0.5109 ∞

$(-\infty, 0.5109) \Rightarrow f(x)$ is concave down

$(0.5109, \infty) \Rightarrow f(x)$ is concave up

You'll also need to be able to identify the point(s) where concavity changes. A point where concavity changes is called a ***point of inflection***.

You can use the same number line that you created to determine concavity intervals to find the *x* coordinate of any inflection points. If there is a sign change across a number in the domain of the function, then you have a point of inflect at that x-value.

Example 9: Find any inflections points of the function in Example 8.

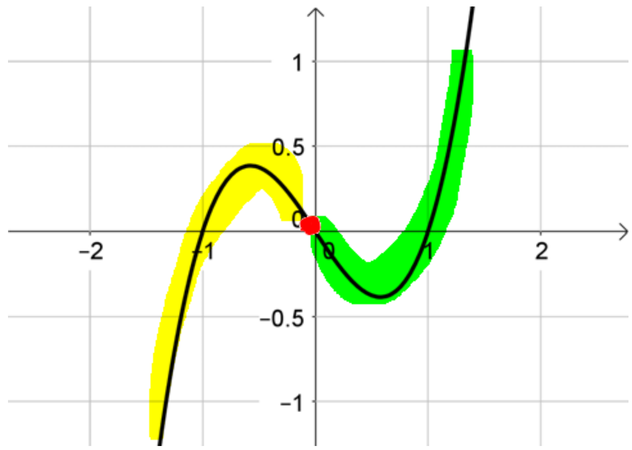
$(0.5109, -5.2438)$

$f(.5109)$

GGB can give us any points of inflection of a polynomial function. The command is:
inflectionpoint[<polynomial>]

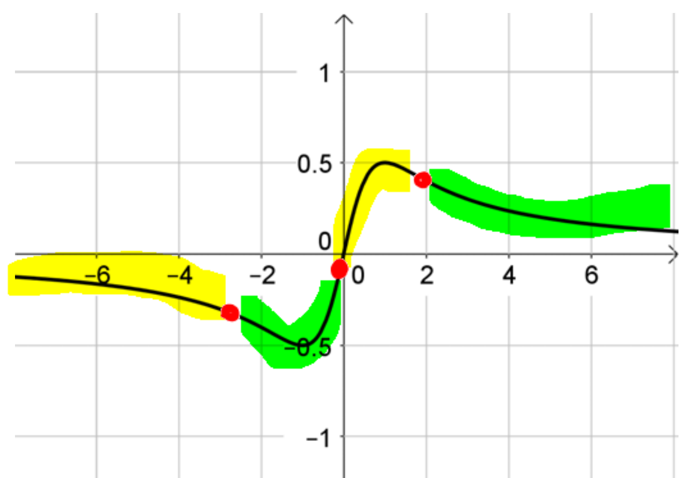
Analyzing a Function

Example 10: The graph given below is the graph of the polynomial function, state whether each of the statements below is true or false.



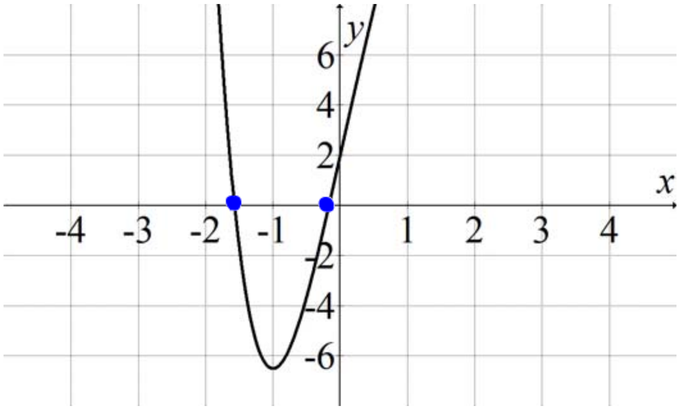
I. The function is concave up over one interval and concave down over one interval. True
II. The function has two points of inflection. False

Example 11: The graph given below is the graph of the polynomial function, state whether each of the statements below is true or false.



I. The function is concave up over two intervals and concave down over ~~tow~~ two intervals. True
II. The function has three points of inflection. True

Example 12: Analyze the function: $f(x) = \frac{3}{2}x^4 - 2x^3 + 12x + 2$



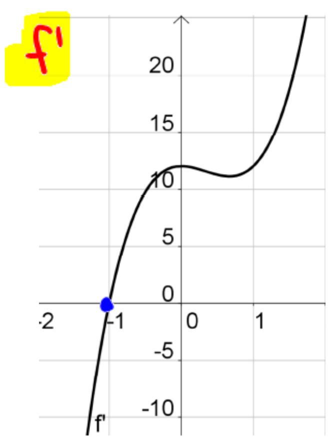
a. Domain $(-\infty, \infty)$

b. Coordinates of any zeros.

Command: root[f]

Answer: $(-1.5695, 0)$ & $(-0.1675, 0)$

c. Interval(s) on which the function is increasing; interval(s) on which the function is decreasing.



Command: $f'(x)$

root[f'] $x = -1$

f: ↘ ↗

f': − +

$-\infty$ -1 ∞

Increasing: $(-1, \infty)$

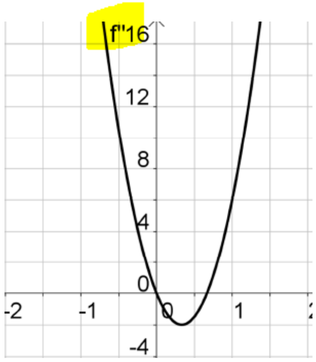
Decreasing: $(-\infty, -1)$

d. Coordinates of any relative extrema.

$(-1, f(-1))$ $\boxed{(-1, -6.5)}$ Rel. min

OR extremum[f]

e. Interval(s) on which the function is concave upward; interval(s) on which the function is concave downward.



Command: f''

root $[f'']$ (0,0) & (0.6667,0)

f : ∪ ∩ ∪

f'' : + − +

$-\infty$ 0 0.6667 ∞

Concave Up: $(-\infty, 0) \cup (0.6667, \infty)$

Concave Down: $(0, 0.6667)$

f. Coordinates of any inflection points.

Command: inflectionpoint [f]

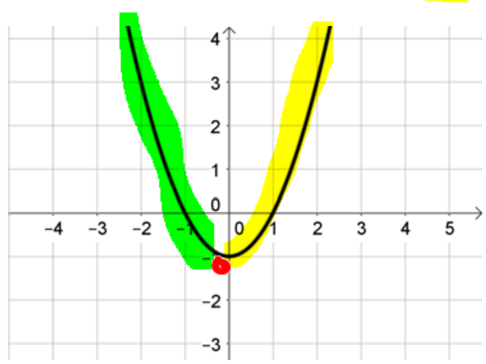
Answer: $f(0)$ $f(0.6667)$

(0, 2) & (0.6667, 9.7037)

We may also find intervals of concavity by simply using the first derivative.

Fact: The function f is concave upward on the interval(s) where f' is increasing. The function f is concave downward on the interval(s) where f' is decreasing.

Example 13: The graph below is the graph of the first derivative of a polynomial function.



a. Find the interval(s) on which the function is concave upward.

$(0, \infty)$

b. Find the interval(s) on which the function is concave downward.

$(-\infty, 0)$

c. Find the x coordinate of any inflection points. $x = 0$