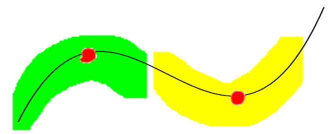
Math 1314
Lesson 15
Second Derivative Test and Optimization

There is a second derivative test to find relative extrema. It is sometimes convenient to use; however, it can be inconclusive. Later in the course, we will use a similar second derivative test to find maxima and minima of functions with two variables.

The Second Derivative Test

1. Find all critical numbers. (Recall: Critical numbers are in the domain of the function where $f'(x)=0$ or where $f'(x)$ is undefined.)

For example:



2. Compute $f''(c)$ for each critical number c.
   (a) If $f''(c) > 0$, then f has a relative minimum at c.
   (b) If $f''(c) < 0$, then f has a relative maximum at c.
   (c) If $f''(c) = 0$, then the test fails. It is inconclusive. Try another method.

Note: To find the y coordinate of a relative maximum or relative minimum, substitute the value you found for x into the original function.

Example 1: Find the x and y coordinates of any relative extrema using the Second Derivative Test if $f(x) = \frac{1}{3}x^3 - 2x^2 - 5x - 10$.

$f'(x) = x^2 - 4x - 5$
$= (x-5)(x+1)$

root [f']

Critical pts: $x = 5$, $x = -1$

$f''(x) = 2x - 4$

$f''(5) = 6 > 0$ Rel. min at $x=5$
$f(5) = -43.3333 = y$

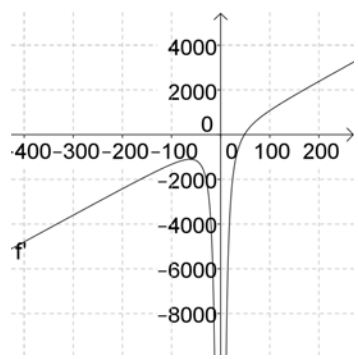
$f''(-1) = -6 < 0$ Rel max at $x=-1$
$f(-1) = -7.3333 = y$

Now you'll work some problems where the objective is to optimize a function. That means you want to make it as large as possible or as small as possible depending on the problem.

1. Find the derivative of the function.
2. Find any critical points.
3. Once you have the critical points, use the second derivative test to verify that you have an absolute max or an absolute min.

Example 2: A company finds that the cost to produce x units of its best-selling item can be modeled by the function $C(x)=\dfrac{1,411,788}{x}+6x^2$. How many units should the company produce if it wishes to minimize its cost? What is the minimum cost?
Enter the function into GGB then find its domain. Domain : $(0, \infty)$

1. Find its derivative using GGB.
Command: C'



2. Find any critical points.
Command: roots[C', 0, 100]

Answer: $x = 49$ ← critical #

3. Verify you have a minimum.
Command: $C''(49) = 36 > 0$ Rel & abs min

Answer:

NOTE
0 is not a critical # since 0 is not in the domain

4. How many units should the company produce if it wishes to minimize its cost?
49 units

$C(49) = 43218$

Min cost \$43,218

If the function is not given, the first task is to write a function that describes the situation in the problem.

1. Read the problem carefully to determine what function you are trying to find.
2. If possible, draw a picture of the situation. Choose variables for the values discussed and put them on your picture.
3. Determine if there are any formulas you need to use, such as area or volume formulas.

Example 3: A homeowner wants to fence in a rectangular vegetable garden using the back of her garage (which measures 20 feet across) as part of one side of the garden. She has 110 feet of fencing material and wants to use that to build the fence. What should be the dimensions of the garden if she fences in the maximum possible area? What is the maximum area?

MAX

1. Determine the function that describes the situation, and write it in terms of one variable (usually *x*).

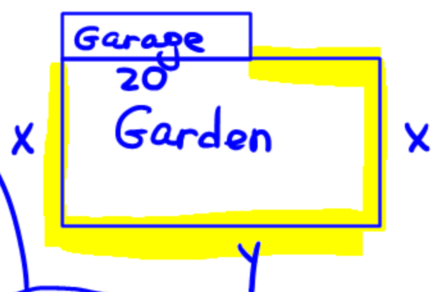
$\text{Area} = xy$

$110 = x + y + x + y - 20$

$110 = 2x + 2y - 20$

$130 = 2x + 2y$ $\quad 65 = x + y$

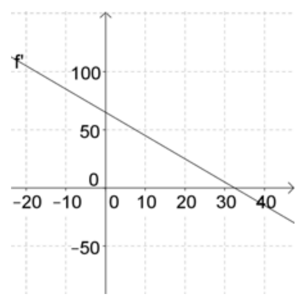
$y = 65 - x$



2. Find its derivative using GGB.
Command:

$A(x) = x(65 - x)$

$A'(x) = -2x + 65$



2. Find any critical points.
Command: Answer:

root [A'] $\quad x = 32.5 \Leftarrow$ critical point

3. Verify you have a maximum.
Command: Answer:

$A''(32.5) = -2 < 0$ $\quad$ Abs. max

4. Dimensions?

$x = 32.5\text{ ft} \Leftarrow$ width

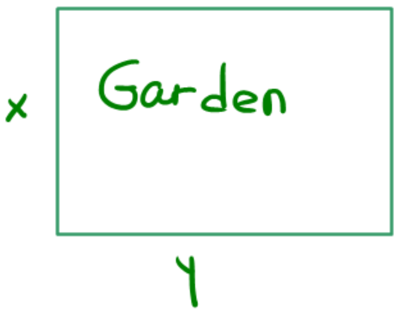
$y = 65 - x = 65 - 32.5 = 32.5\text{ ft} \Leftarrow$ length

5. Max area?
Command: Answer:

$A(32.5)$ $\quad 1056.25\text{ ft}^2$

Example 4: A man wishes to have a vegetable garden enclosed by a fence in his backyard. The garden is to be a rectangular area of 289 ft ^{2}. Find the dimensions of the garden that will minimize the amount of fencing material needed.

1. Determine the function that describes the situation, and write it in terms of one variable (usually *x*).



$A = xy$ $\quad 289 = xy$

Solve for y

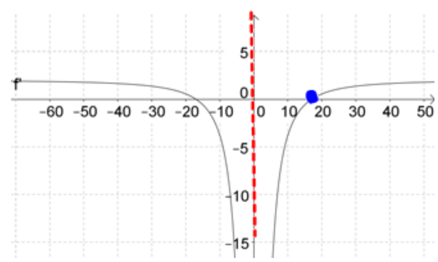
MIN $P = 2x + 2y$ $\quad y = \frac{289}{x}$

$P(x) = 2x + 2\left(\frac{289}{x}\right)$

$P(x) = 2x + \frac{578}{x}$ $\quad$ Domain: $(0, \infty)$

2. Find its derivative using GGB.
Command:

$P' = \frac{2x^2 - 578}{x^2}$



2. Find any critical points.
Command: roots[P', 10, 20]

Answer: $x = 17 \Leftarrow$ critical point

NOTE $x = 0$ is NOT a critical #, since $x = 0$ is NOT in the Domain of P

3. Verify you have a minimum.
Command: $P''(17) = 0.2353 > 0$

Answer: Abs. min

4. Dimensions?

$x = 17$ ft

$y = \frac{289}{x} = \frac{289}{17} = 17$ ft

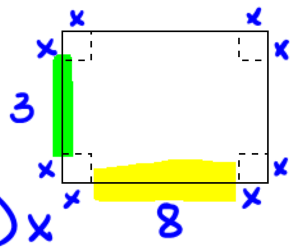
$\boxed{17\text{ft} \times 17\text{ft}}$

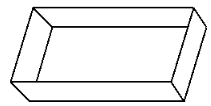
Example 5: If you cut away equal squares from all four corners of a piece of cardboard and fold up the sides, you will make a box with no top. Suppose you start with a piece of cardboard the measures 3 feet by 8 feet. Find the dimensions of the box that will give a maximum volume. What is the maximum volume?

MAX

1. Determine the function that describes the situation, and write it in terms of one variable (usually *x*).

$V = lwh$

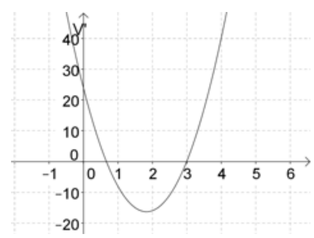




$V(x) = (8-2x)(3-2x)x$

2. Find its derivative using GGB.
Command:

$V' = 12x^2 - 44x + 24$



2. Find any critical points.
Command: root[V']

Answer:

$x = .6667$

$\cancel{x = 3}$

3. Verify you have a maximum.
Command:

Answer:

$V''(.6667) = -27.9992 < 0$ Abs. max

4. Dimensions?

$x = .6667$ ft ⇐ height

$3 - 2x = 3 - 2(.6667) = 1.6667$ ft ⇐ width

$8 - 2x = 8 - 2(.6667) = 6.6667$ ft ⇐ length

6.6667 ft × 1.6667 ft × .6667 ft

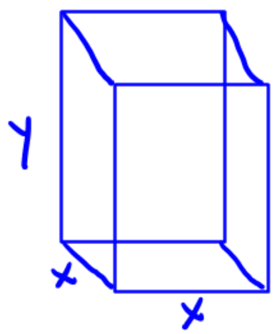
5. Max volume?
Command:

Answer:

$V(.6667) = 7.4074\ ft^3$

Example 6: Postal regulations state that the girth plus length of a package must be no more than 104 inches if it is to be mailed through the US Postal Service. You are assigned to design a package with a square base that will contain the maximum volume that can be shipped under these requirements. What should be the dimensions of the package? (Note: girth of a package is the perimeter of its base.)

1. Determine the function that describes the situation, and write it in terms of one variable (usually *x*). MAX



$$V = x^2 y$$

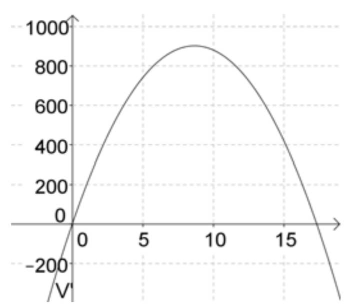
$$104 = 4x + y \quad \text{Solve for } y$$

$$y = 104 - 4x$$

$$V(x) = x^2(104 - 4x)$$

2. Find its derivative using GGB.
Command:

$$V' = -12x^2 + 208x$$



2. Find any critical points.
Command: root[V']

Answer:

$\cancel{x = 0}$

$x = 17.3333$

3. Verify you have a maximum.
Command:

Answer:

$V''(17.3333) = -207.9992 < 0$ Abs. max

4. Dimensions?

$x = 17.3333$ inches ⇐ width & height

$y = 104 - 4x = 104 - 4(17.3333) = 34.6667$ ⇐ length

Try this one: An open box has a square base and a volume of $500\ \text{in}^3$. Find the dimensions of the box, assuming a minimum amount of material is used in its construction.

Try this one: You are assigned to design some shipping materials at minimum cost. The package will be a closed rectangular box with a square base, and must have a volume of 50 cubic inches. The material used for the top costs 35 cents per square inch, the material used for the bottom of the box costs 45 cents per square inch, and the material used for the sides costs 20 cents per square inch. Find the dimensions of the box that will minimize the cost. What is the minimum cost?