Math 1314
Lesson 1: Prerequisites

1. Exponents

Recall: $x^{-n} = \frac{1}{x^n}$ $\qquad x^{m/n} = \sqrt[n]{x^m}$

Example 1: Simplify and write the answer without using negative exponents:

a. $2x^{-5}$ $= 2 \cdot \frac{1}{x^5} = \frac{2}{x^5}$

b. $(2x)^{-5}$ $= \frac{1}{(2x)^5} = \frac{1}{2^5 x^5} = \frac{1}{32x^5}$

Example 2: Write as a radical: $x^{\frac{3}{4}}$ $= \sqrt[4]{x^3}$

Example 3: Write using a rational exponent: $\sqrt[3]{x^5}$ $= x^{\frac{5}{3}}$

2. *Identifying Polynomials*

To begin with, let's review the definition of a polynomial function.

A **polynomial** is the sum and/or difference of terms that contain variables and/or real constants, with variables raised to whole number (0, 1, 2, 3, …) powers.

Example 4: Which of the following are polynomial functions?

a. $f(x) = -2x^5 + 0.5x + \frac{3}{2}$ YES

b. $f(x) = \sqrt{3}x^3 - 2x^{-3}$ NO

c. $f(x) = \frac{1}{x^4} - 10x$ $= x^{-4} - 10x$ NO

d. $f(x) = x^3 - 2x^2 + \frac{3}{2}x - \sqrt{5}$ YES

e. $f(x) = 5x^{0.5} - 0.5x^{1/8}$ NO

f. $f(x) = \frac{x^6 - e}{x + 2}$ NO

g. $f(x) = \ln(x^2 - 3)$ NO

h. $f(x) = 3e^{3x} - 1$ NO

i. $f(x) = -5$ YES $-5x^0$

Not allowed

x^{-2} or $\frac{1}{x^2}$ $\qquad |x|$

$\sqrt{x}$ or $x^{1/2}$ $\qquad \ln x$

e^x

3. Simplifying, Solving and Evaluating Polynomials

Example 5: Simplify $-2x^2(-3x+4)-x(4x+x^2-1)+5$

$= 6x^3 - 8x^2 - 4x^2 - x^3 + x + 5$

$= \boxed{5x^3 - 12x^2 + x + 5}$

Example 6: Multiply $(x-1)(2x+3)$ FOIL

$= 2x^2 + 3x - 2x - 3 = \boxed{2x^2 + x - 3}$

The solution(s) (root(s), zero(s), x-intercept(s)) of a polynomial function $f(x)$ is/are found by finding the values of the variable x when $f(x)=0$. $a^2 - b^2 = (a-b)(a+b)$

Example 7: Find the roots of the function:

a. $f(x)=x^2-2x-3$

$(x-3)(x+1)=0$

$x-3=0$ $x+1=0$

$\boxed{x=3}$ $\boxed{x=-1}$

b. $f(x)=49x^2-4$

$(7x)^2 - 2^2$

$= (7x-2)(7x+2) = 0$

$7x-2=0$ $7x+2=0$

$7x=2$ $\boxed{x=2/7}$ $7x=-2$ $\boxed{x=-2/7}$

To find the y-intercept of a polynomial function, calculate f(0). *In fact, to calculate the y-intercept of ANY function, calculate f(0).*

Example 8: Find the y-intercept of $f(x)=2x-3$.

$f(0) = 2(0) - 3 = \boxed{-3}$ $\boxed{(0,-3)}$

Example 9: Find f(-2) for each function below, if possible.

a. $f(x)=x^3+10x$

$f(-2) = (-2)^3 + 10(-2)$

$= -8 - 20 = \boxed{-28}$

b. $h(x)=\dfrac{x-10}{x^2-4}$

$h(-2) = \dfrac{-2-10}{(-2)^2-4} = \dfrac{-12}{4-4}$

Does not exist

Is not defined at $x=-2$

$(-2)^2 = (-2)(-2) = 4$

$-2^2 = -4$

Example 10: Let $f(x)=x^2-3$, calculate $\frac{f(6)-f(1)}{6-1}$. $= \frac{33-(-2)}{5} = \frac{35}{5}$

$f(6) = 6^2 - 3$
$= 36-3$
$= 33$

$f(1) = 1^2 - 3$
$= 1-3$
$= -2$

$= \boxed{7}$

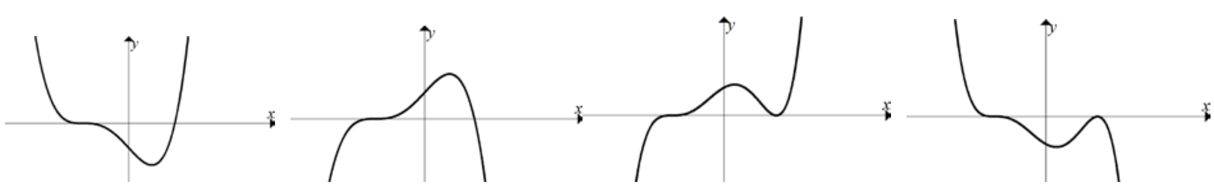
Example 11: Suppose the total cost in dollars to produce x items is given by the function $C(x)=0.0003x^3+0.14x^2+12x+1400$. Find the total cost of producing 50 items.

$C(50) = 0.0003(50)^3 + 0.14(50)^2 + 12(50) + 1400$

$= \$2387.5$

4. Analyzing Graphs of Polynomials

The graph of a polynomial function looks similar to one of the four graphs below.



Notice that the graphs are nice smooth curves, no sharp corners, no holes and no asymptotes.

The **domain** of a function is the set of all valid inputs. The domain of any polynomial function is $(-\infty,\infty)$, which we can see clearly from the graphs (the set of x-values on the graph).

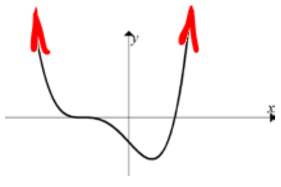
The **range** of a function is the set of output given valid inputs. The range is best found by observing the graph (set of y-values on the graph).

The **end behavior** of a polynomial function is the behavior of the polynomial to the far left and far right of the graph. If we are given only the function and not the graph, we can determine the end behavior by simply looking at its leading term (term with the highest power on the variable x).

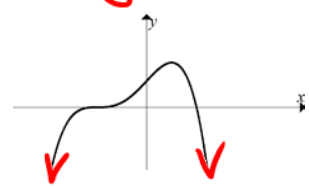
End Behavior of a Polynomial Function

An even-degree polynomial's end behavior will be $\uparrow\uparrow$ if its leading coefficient is positive or $\downarrow\downarrow$ if its leading coefficient is negative.

positive

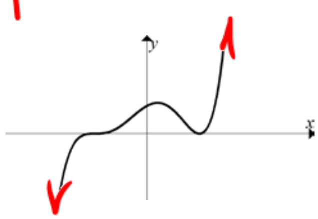


negative

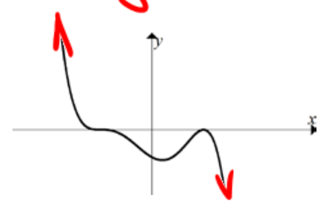


An odd-degree polynomial's end behavior will be $\downarrow\uparrow$ if its leading coefficient is positive or $\uparrow\downarrow$ if its leading coefficient is negative.

positive



negative



Example 12: Determine the end behavior of $f(x) = -3x^5 + 2x$. What is its range?

L.C. – negative

Degree = 5 = odd $\quad \uparrow\downarrow$

Example 13: Determine the end behavior of $f(x) = -(3x+5)^2(2x-1)^2$.

Leading Term $= -(3x)^2(2x)^2 = -9x^2(4x^2)$

$= -36x^4$

L.C. → negative

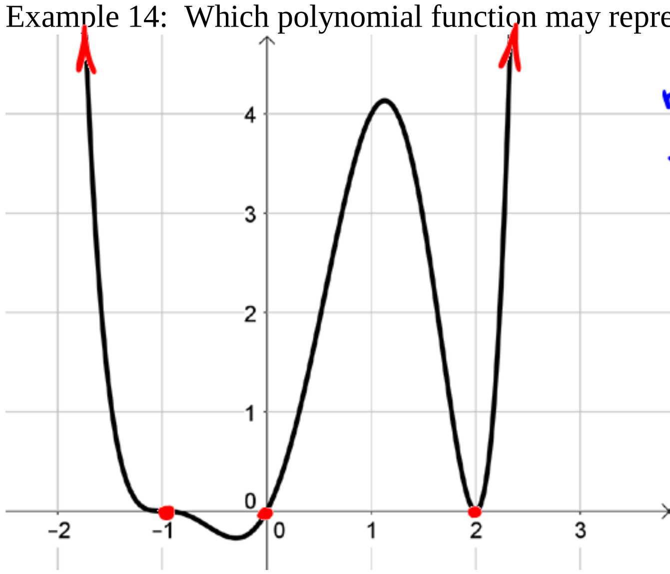
Degree = 4 = even $\quad \downarrow\downarrow$

Description of the Behavior at Each *x*-intercept

1. Even Multiplicity: The graph touches the *x*-axis, but does not cross it (looks like a parabola there). $f(x) = (x-1)^2$
2. Odd Multiplicity of 1: The graph crosses the *x*-axis (looks like a line there). $f(x) = x - 1$
3. Odd Multiplicity greater than or equal to 3: The graph crosses the *x*-axis and it looks like a cubic there. $f(x) = (x-1)^3$

1

Example 14: Which polynomial function may represent the given graph?



a root	$x-a$ factors
-1	$x-(-1) = (x+1)^3$
0	x^1
2	$(x-2)^2$

↑↑ L.C. = (+)
even degree

a. $f(x) = 0.5(x+2)(x-1)$

b. $f(x) = -0.5x(x+2)^2(x-1)^3$

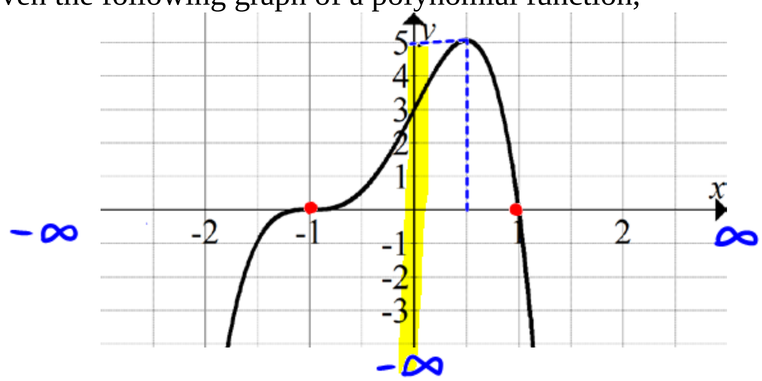
→ c. $f(x) = -0.5x(x-2)^2(x+1)^3$

d. $f(x) = 0.5x(x+2)^2(x-1)^3$

→ e. $\boxed{f(x) = 0.5x(x-2)^2(x+1)^3}$

Other times we're given the graph of a polynomial function and are asked to find certain values or when the function is positive or negative.

Example 15: Given the following graph of a polynomial function,



a. For which x-value(s) is the function equal to 0.

$x=-1$, $x=1$ $\quad f(-1)=0 \quad f(1)=0$

b. Find f(0.5). $= 5$

c. Find f(0). $= 3$

d. Give the inteval(s) over which the function is negative

$(-\infty,-1) \cup (1,\infty)$

e. Over the interval(s) over which the function is positive.

$(-1,1)$

f. Find the domain and the range.

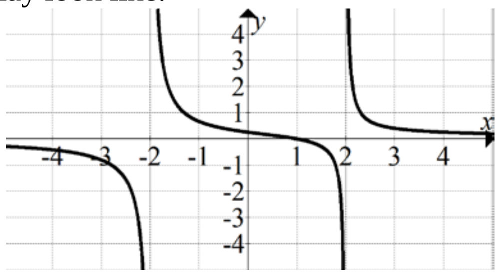
D: $(-\infty,\infty)$

R: $(-\infty,5]$

5. Rational Functions

If we form the ratio of two polynomials we obtain a **rational function**.

Graphs of rational function may look like:



These types of graphs may have zeros, no more than one y-intercept, vertical and horizontal asymptotes, and/or holes.

The **domain** of such a rational function is all real numbers except those that make the denominator equal to zero.

Example 16: Find the domain of $f(x) = \dfrac{2}{x^2 - 3x}$.

$x^2 - 3x \neq 0$

$x(x-3) \neq 0$

$x \neq 0 \quad x - 3 \neq 0$

$x \neq 3$

$-\infty \quad 0 \quad 3 \quad \infty$

$(-\infty, 0) \cup (0, 3) \cup (3, \infty)$

The roots are where the graph crosses the x-axis. To find the roots, simplify the function then set the numerator equal to zero and solve for x.

$a^2 - b^2 = (a-b)(a+b)$

Example 17: Find any roots of $g(x) = \dfrac{x^2 + 2x - 48}{x^2 - 36}$. $= \dfrac{(x+8)\cancel{(x-6)}}{\cancel{(x-6)}(x+6)}$

$x + 8 = 0$

$\boxed{x = -8}$

A vertical line is a **vertical asymptote** of a rational funciton if its graph approaches that line at the far top and far bottom of the graph. *The graph can never cross these lines.*

To find a rational function's roots, vertical asymptotes and holes you must first factor the numerator and denominator as much as possible and simplify. Then, LOOK at the denominator.

- If a factor cancels with a factor in the numerator, then there is a hole where that factor equals zero.
- If a factor does not cancel, then there is a vertical asymptote where that factor equals zero.

Example 18: Find any vertical asymptotes of $g(x)=\dfrac{x^2+3x-10}{x^2-25}$. $= \dfrac{(x+\cancel{5})(x-2)}{(x-5)(x+\cancel{5})}$

Holes: $x+5=0$ $\quad x=-5$ $\quad \boxed{(-5, .7)}$ $\quad \dfrac{-5-2}{-5-5} = \dfrac{-7}{-10} = 7/10 = .7$

Vert. Assym: $x-5=0$ $\quad \boxed{x=5}$

A horizontal line is a **horizontal asymptote** of a rational funciton if its graph approaches that line at the far left and far right of the graph. *The graph may cross this line*.

Shorthand: degree of f = deg(f), numerator = N, denominator = D

Let $f(x)=\dfrac{p(x)}{q(x)}$,

1. If deg(N) > deg(D) then there is no horizontal asymptote.
2. If deg(N) < deg(D) then there is a horizontal asymptote and it is $y = 0$ (x-axis).
3. If deg(N) = deg(D) then there is a horizontal asymptote and it is $y = \dfrac{a}{b}$, where

    a is the leading coefficient of the numerator.
    b is the leading coefficient of the denominator.

Example 19: Find any horizontal asymptote given:

a. $g(x)=\dfrac{5x+25}{x^2-7x+12}$

deg(N) < deg(D)
1 < 2

$y=0$

b. $g(x)=\dfrac{-10x^2+1}{15x^2-3}$

deg(N) = deg(D)
2 = 2

$y = \dfrac{-10}{15} = -2/3$ $\quad \boxed{y=-2/3}$

c. $f(x) = \frac{5}{x^2 - 3}$ $\deg(N) < \deg(D)$

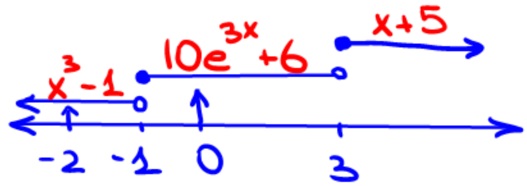
$0 < 2$

6. *Piecewise Defined Functions*

$y = 0$

A function that is defined by two (or more) equations over a specified domain is called a **piecewise function**.

Example 20: Let $f(x) = \begin{cases} x^3 - 1, & x < -1 \\ 10e^{3x} + 6, & -1 \le x < 3 \\ x + 5, & x \ge 3 \end{cases}$.



Find:

a. $f(0)$.

$= 10\cancel{e^{3\cdot 0}}^{\,1} + 6$

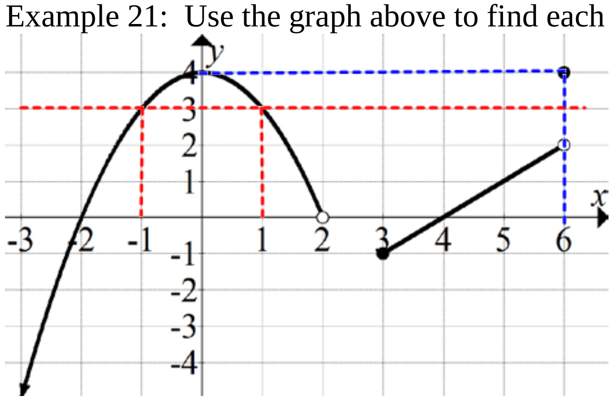
$= 16$

b. $f(-2)$.

$= (-2)^3 - 1$

$= -8 - 1 = -9$

Example 21: Use the graph above to find each of the following.



a. $f(3)$ $= -1$

b. $f(6)$ $= 4$

c. $f(2)$ DNE

d. For which x-value(s) is f(x) = 3.

$x = -1$

$x = 1$

The last type of function whose domain we need to review is the square root function.

Recall that over real numbers we cannot take any even root of a negative number. Hence, to find the its domain, exclude real numbers that result in an even root of a negative number.

Example 22: Find the domain and write the answer in interval notation:

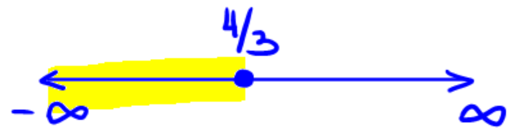
a. $f(x) = \sqrt{-3x+4}$

$-3x+4 \geq 0$

$\dfrac{-3x}{-3} \geq \dfrac{-4}{-3}$

flip

$x \leq 4/3$



$(-\infty, 4/3]$

b. $f(x) = \sqrt{5x+1}$

$5x+1 \geq 0$

$\dfrac{5x}{5} \geq \dfrac{-1}{5}$

$x \geq -1/5$

$[-1/5, \infty)$

Now you can take Practice Test 1 (up to 20 times) then take Test 1 (up to 2 times) from anywhere online (no CASA reservation needed).