**Math 1314**
**Lesson21**
**Other Applications of Integration**

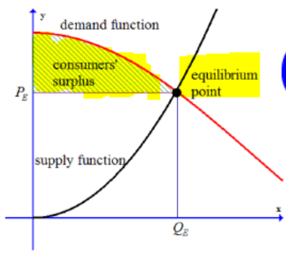
## Consumers' Surplus and Producers' Surplus

*An example: A store has a 2-L soda on sale for 89 cents each.* C.S.

*You go to the store and are prepared to pay $1.29, but you save 40 cents on each bottle.*
*The store bought each for 58 cents, so it's coming out ahead by 31 cents on each bottle.*
P.S.

*Surplus is how much each party gains from the transaction.*

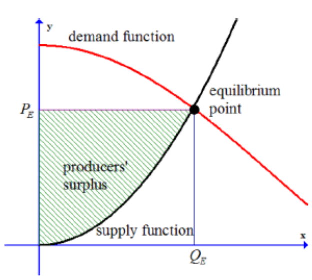
The **consumers' surplus** is defined to be the difference between what customers would be willing to pay and what they actually pay. It is the area of the region bounded below by the demand function and above by the line that represents the unit market price. In the sketch shown below, the shaded region represents the consumers' surplus.




$(Q_E, P_E)$
quantity price

Then the consumers' surplus is given by $CS = \int_0^{Q_E} D(x)\,dx - Q_E \cdot P_E$. In this formula, $D(x)$ represents the demand function, $Q_E$ represents the quantity sold and $P_E$ represents the price.

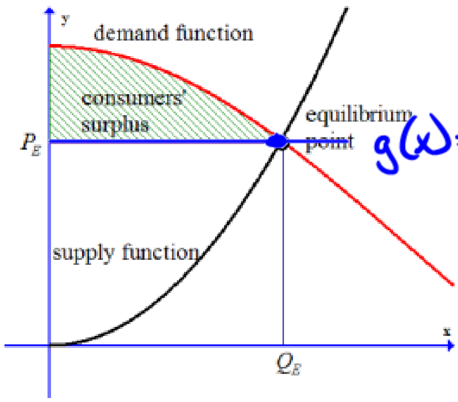
Similarly, producers may be willing to sell their product for a lower price than the prevailing market price. If the market price is higher than where producers expect to price their items, then the difference is called the **producers' surplus**. The producer's surplus is the area of the region bounded below by the line that represents the price and above by the supply curve.
In the sketch shown below, the shaded region represents the producers' surplus.




The producers' surplus is given by $PS = Q_E \cdot P_E - \int_0^{Q_E} S(x)\,dx$. In this formula, $S(x)$ represents the supply function, $P_E$ represents the unit market price and $Q_E$ represents the quantity supplied.

Example 1: Suppose the demand for a certain product is given by $p = D(x) = -0.01x^2 - 0.1x + 6$ where $p$ is the unit price given in dollars and $x$ is the quantity demanded per month given in units of 1000. The market price for the product is \$4 per unit. $= P_E$

*Recall*:




$g(x) = 4$

a. Find the quantity demanded.

Command:

intersect [D, g]

[D, 4]

Answer: $Q_E$ $P_E$

$\cancel{(-20, 4)}$ & $(10, 4)$

Quantity Demanded:

b. Find the consumers' surplus if the market price for the product is \$4 per unit.

*Recall*: $CS = \int_0^{Q_E} D(x)\,dx - Q_E \cdot P_E$

First apply the formula.

$$CS = \int_0^{10} \underbrace{(-0.01x^2 - 0.1x + 6)}_{D(x)}\,dx - \underbrace{(10)}_{Q_E}\underbrace{(4)}_{P_E}$$

Command:

integral [D, 0, 10] - 10*4

Answer:

11.6667 * 1000

\$ 11,666.7

Sometimes, the unit price will not be given. Instead, product will be sold at market price, and you'll be given both supply and demand equations. In this case, we can find the equilibrium point (Section 4.2) which will give us the equilibrium quantity and price.

Example 2: The demand function for a product is given by $p = D(x) = -0.4x + 23$ where *p* is the unit price in dollars and *x* is the quantity demanded. The supply function for the same product is given by $p = S(x) = 0.03x^2 + 3$ where *p* is the unit price in dollars and *x* is the quantity supplied. Find the equilibrium point. Then find both the consumers' and producers' surplus at the market price.

*Begin by entering the functions into GGB.*

a. Assume the market price is set at the equilibrium price. Find the equilibrium point.

*Recall: The equilibrium point is where the demand and supply curve intersect.*

Command: intersect [D, S]

Answer: ~~(-33.3333, 36.3333)~~

(20, 15)

$Q_E$ $P_E$

Equilibrium Quantity: 20 units $= Q_E$

Equilibrium Price: \$15 $= P_E$

b. Determine the consumers' surplus.

*Recall:* $CS = \int_0^{Q_E} D(x)\,dx - Q_E \cdot P_E$

First apply the formula.

$$CS = \int_0^{20} \underbrace{(-0.4x + 23)}_{D}\,dx - \underbrace{20}_{Q_E} * \underbrace{15}_{P_E}$$

Command: integral [D, 0, 20] - 20 * 15

Answer: \$80

c. Determine the producers' surplus.

*Recall:* $PS = Q_E \cdot P_E - \int_0^{Q_E} S(x)\,dx$

First apply the formula.

$$PS = 20 * 15 - \int_0^{20} (0.03x^2 + 3)\,dx$$

Command: 20 * 15 - integral [S, 0, 20]

Answer: \$160

**Probability**

The study of probability deals with the likelihood of a certain outcome of an experiment.

We will be interested in finding the probability of something occurring over a continuous interval. Suppose you want to know how long a brand of light bulb lasts. Now the possible set of answers contains more than a discrete set of numbers (e.g., something other than 1, 2, 3, etc.). The light bulbs can last any positive length of time. We can find the probability that the light bulbs life span is on a given interval [a, b].

This problem is an example of a problem that involves a probability density function. This type of function can be used to determine the probability that an event occurs on a given interval [a, b]. All probability density functions must meet these three criteria:

1. $f(x) \geq 0$ for all $x$
2. The area under the graph of $f(x)$ is exactly 1.
3. The probability that an event occurs in an event [a, b] can be computed using the definite integral $\int_a^b f(x)\,dx$.

Example 3: The function $f(x) = 0.002e^{-0.002x}$ gives the life span of a popular brand of light bulb, where $x$ gives the lifespan in hours and $f(x)$ is the probability density function. Find the probability that the lifespan is between 500 hours and 1000 hours.

a. Set up the integral needed to answer the question.

$$\int_{500}^{1000} 0.002e^{-0.002x}\,dx$$

b. Find the probability.

Command: Answer:

integral[f, 500, 1000] 0.2325

Example 4: A company finds that the percent of its locations that experience a profit in the first year of business has the probability density function $P(x) = \frac{36}{11}x\left(1 - \frac{1}{3}x\right)^2$, $0 \le x \le 1$.

0.3 0.6

a. What is the probability that between 30% and 60% of the company's locations experienced a profit during the first year of business?

i. Set up the integral needed to answer the question.

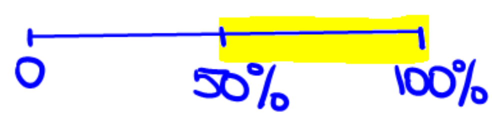
$$\int_{0.3}^{0.6} \frac{36}{11}x\left(1 - \frac{1}{3}x\right)^2 dx$$

ii. Find the probability.

Command: integral[P,0.3,0.6]

Answer: .3154

b. What is the probability that more than 50% of the company's locations experienced a profit during the first year of business?




i. Set up the integral needed to answer the question.

$$\int_{0.5}^{1} \frac{36}{11}x\left(1 - \frac{1}{3}x\right)^2 dx$$

ii. Find the probability.

Command: integral[P,0.5,1]

Answer: 0.6761