Math 1314
Lesson 23
Partial Derivatives

When we are asked to find the derivative of a function of a single variable, $f(x)$, we know exactly what to do. However, when we have a function of two variables, there is some ambiguity. With a function of two variables, we can find the slope of the tangent line at a point *P* from an infinite number of directions. We will only consider two directions, either parallel to the *x* axis or parallel to the *y* axis. When we do this, we fix one of the variables. Then we can find the derivative with respect to the other variable.

So, if we fix *y*, we can find the derivative of the function with respect to the variable *x*. And if we fix *x*, we can find the derivative of the function with respect to the variable *y*.

These derivatives are called **partial derivatives**.

First-Order Partial Derivatives

We will use two different notations:

$\frac{\partial f}{\partial x} = f_x$ In this case, you consider *y* as a constant.

$\frac{\partial f}{\partial y} = f_y$ In this case, you consider *x* as a constant.

We can use GGB to determine the first-order partial derivatives. The command is:
derivative[<function>,<variable>]

Example 1: Suppose $f(x,y) = x^2 - 3xy^2 + 4y^2$. *Enter the function into GGB.*

a. Find $\frac{\partial f}{\partial x} = f_x$. $f_x = 2x - 3y^2$ $(cy^2)' = c \cdot 2y$

Command: derivative [f, x]

Answer: $f_x = -3y^2 + 2x$

b. Find $\frac{\partial f}{\partial y} = f_y$. $f_y = -6xy + 8y$

Command: derivative [f, y]

Answer: $f_y = -6xy + 8y$

We can also evaluate the first partial derivatives at a given point.

Example 2: Find the first-order partial derivatives of the function $f(x,y)=4x^3y^2+2x^2y^3-12x^2+3y^2+10$ evaluated at the point (-1, 3).
Enter the function into GGB.

a. $f_x\big|_{(-1,3)}$

Command: derivative[f, x]

Answer: $a(x,y)=12x^2y^2+4xy^3-24x$

Command: a(-1,3)

Answer: 24

b. $f_y\big|_{(-1,3)}$

Command: derivative [f, y]

Answer: $c(x,y)=8x^3y+6x^2y^2+6y$

Command: c(-1,3)

Answer: 48

Second-Order Partial Derivatives

Sometimes we will need to find the second-order partial derivatives. To find a second-order partial derivative, you will take respective partial derivatives of the first partial derivative. There are a total of 4 second-order partial derivatives.

There are two notations, but we will only use one of them.

$$f_{xx}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad f_{xy}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$\neq$

$$f_{yy}=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \qquad f_{yx}=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Example 3: Evaluate the second-order partial derivatives of $f(x, y) = 3x^2 - x^3y^3 + 5xy + 6y^3$ at the point (1, 2). *Enter the function into GGB. Then produce the first-order partials.*

a. f_x

Command: Answer:

derivative[f, x]

f_x

$a(x,y) = -3x^2y^3 + 6x + 5y$

b. f_y

Command: Answer:

derivative[f, y]

f_y

$b(x,y) = -3x^3y^2 + 5x + 18y^2$

c. $f_{xy}\big|_{(1,2)}$

Commands: Answers:

derivative[a, y]

f_{xy}

$c(x,y) = -9x^2y^2 + 5$

c(1,2)

-31

d. $f_{yy}\big|_{(1,2)}$

Commands: Answers:

derivative[b, y]

$e(x,y) = -6x^3y + 36y$

e(1,2)

60

A function of the form $f(x, y) = ax^b y^{1-b}$ where a and b are positive constants and $0 < b < 1$ is called a ***Cobb-Douglas production function***. In this function, x represents the amount of money spent for labor, and y represents the amount of money spent on capital expenditures such as factories, equipment, machinery, tools, etc. The function measures the output of finished products.

The first partial with respect to x is called the **marginal productivity of labor**. It measures the change in productivity with respect to the amount of money spent for labor. In finding the first partial with respect to x, the amount of money spent on capital is held at a constant level.

The first partial with respect to y is called the **marginal productivity of capital**. It measures the change in productivity with respect to the amount of money spent on capital expenditures. In finding the first partial with respect to y, the amount of money spent on labor is held at a constant level.

Example 4: A country's production can be modeled by the function $f(x, y) = 50x^{2/3} y^{1/3}$ where x gives the units of labor that are used and y represents the units of capital that were used.

a. Find the first-order partial derivatives and label each as marginal productivity of labor or marginal productivity of capital.
Enter the function into GGB.

Command: Answer:

derivative[f, x]

$$a(x,y) = \frac{100\sqrt[3]{x^2}\sqrt[3]{y}}{x}$$

marginal productivity of labor

Command: Answer:

derivative[f, y]

$$b(x,y) = \frac{50\sqrt[3]{x^2}\sqrt[3]{y}}{3x}$$

marginal productivity of capital

b. Find the marginal productivity of labor and the marginal productivity of capital when the amount expended on labor is 125 units and the amount spent on capital is 27 units.

x y

Command: Answer:

a(125, 27)

20 units / unit increase of labor

Command: Answer:

b(125, 27)

46.2963 units / unit increase in capital