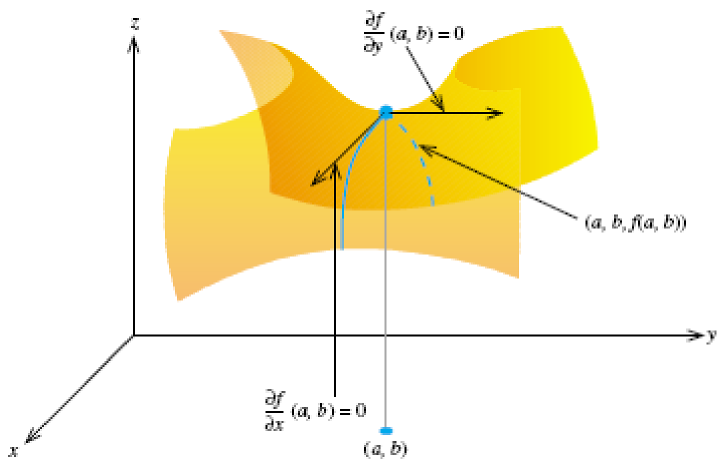
Math 1314
Lesson 24
Maxima and Minima of Functions of Several Variables

We learned to find the maxima and minima of a function of a single variable earlier in the course. We had a second derivative test to determine whether a critical point of a function of a single variable generated a maximum or a minimum, or possibly that the test was not conclusive at that point. We will use a similar technique to find relative extrema of a function of several variables.

Since the graphs of these functions are more complicated, determining relative extrema is also more complicated. At a specific critical number, we can have a max, a min, or something else. That "something else" is called a saddle point.



The method for finding relative extrema is very similar to what you did earlier in the course.

1. Find the first partial derivatives and set them equal to zero. You will have a system of equations in two variables which you will need to solve to find the critical points.

2. Apply the second derivative test. To do this, you must find the second-order partial derivatives. Let $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2$. You will compute $D(a,b)$ for each critical point (a,b). Then you can apply the second derivative test for functions of two variables:

- If $D(a, b) > 0$ and $f_{xx}(a,b) < 0$, then *f* has a relative maximum at (a, b).
- If $D(a, b) > 0$ and $f_{xx}(a,b) > 0$, then *f* has a relative minimum at (a, b).
- If $D(a, b) < 0$, then *f* has neither a relative maximum nor a relative minimum at (a, b) (i.e., it has a saddle point, which is neither a max nor a min).
- If $D(a, b) = 0$, then this test is inconclusive.

Example 1: Find the relative extrema of the function. $f(x,y) = -3x^2 + 2xy - 2y^2 + 14x + 2y - 8$

Begin by entering the function into GGB.

a. Find the first-order partials.

Commands: derivative[f, x]

derivative[f, y]

Answers:

f_x: $a(x,y) = -6x + 2y + 14 = 0$

f_y: $b(x,y) = 2x - 4y + 2 = 0$

b. Find the point of intersection of the equations in part a. These points of intersection are the critical points of the function *f*.

Command: $-6x + 2y + 14 = 0$

$2x - 4y + 2 = 0$

intersect [g, h]

Answer:

$g: -3x + y = -7$

$h: x - 2y = -1$

$\boxed{(3,2)}$

c. Determine $D(x,y) = f_{xx} \cdot f_{yy} - \left(f_{xy}\right)^2$.

Commands:

derivative[a, x]

derivative[b, y]

derivative[a, y]

Answers:

f_{xx}: $c(x,y) = -6$

f_{yy}: $d(x,y) = -4$

f_{xy}: $e(x,y) = 2$

$D(x,y) = (-6)(-4) - (2)^2 = 24 - 4 = 20 > 0$

d. Apply the second derivative test to classify each critical point found in part b.

$D(3,2) = 20 > 0$

Rel. maximum at (3,2)

$f_{xx}(3,2) = -6 < 0$

e. For any maxima point and ~~minima point~~ found in part d, calculate the maxima and minima, respectively.

Command: $f(3,2)$

Answer: 15 ← max

Example 2: Find the relative extrema of the function $f(x,y) = 2x^3 + y^2 - 9x^2 - 4y + 12x - 2$.
Begin by entering the function into GGB.
a. Find the first-order partials.

Commands: Answers:

derivative[f, x] f_x $a(x,y) = 6x^2 - 18x + 12$

derivative[f, y] f_y $b(x,y) = 2y - 4$

b. Find the point of intersection of the equations in part a. These points of intersection are the critical points of the function *f*.

Command: $6x^2 - 18x + 12 = 0$
$2y - 4 = 0$
intersect [c, g]

Answer:
$c: x^2 - 3x = -2$
$g: y = 2$
$(1,2)$ & $(2,2)$

c. Determine $D(x,y) = f_{xx} \cdot f_{yy} - \left(f_{xy}\right)^2$.

Commands: Answers:

derivative[a, x] f_{xx} $d(x,y) = 12x - 18$

derivative[b, x] f_{yy} $e(x,y) = 2$

derivative[a, y] f_{xy} $h(x,y) = 0$

$D(x,y) = (12x - 18)(2) - (0)^2 = 24x - 36$

d. Apply the second derivative test to classify each critical point found in part b.

$D(1,2) = 24(1) - 36 = -12 < 0$ Saddle point at $(1,2)$

$D(2,2) = 24(2) - 36 = 12 > 0$

$f_{xx}(2,2) = 12(2) - 18 = 6 > 0$ Rel. min at $(2,2)$

e. For any maxima point and minima point found in part d, calculate the maxima and minima, respectively.

Command: Answer:

$f(2,2)$ -2

Example 3: Suppose a company's weekly profits can be modeled by the function $P(x, y) = -0.2x^2 - 0.25y^2 - 0.2xy + 100x + 90y - 4000$ where profits are given in thousand dollars and x and y denote the number of standard items and the number of deluxe items, respectively, that the company will produce and sell. How many of each type of item should be manufactured each week to maximize profit? What is the maximum profit that is realizable in this situation?

Begin by entering the function into GGB.

a. Find the first-order partials.

Commands: Answers:

derivative[P, x] P_x $a(x,y) = \frac{-2x-y+500}{5}$

derivative[P, y] P_y $b(x,y) = \frac{-2x-5y+900}{10}$

b. Find the point of intersection of the equations in part a. These points of intersection are the critical points of the function *f*.

Command: $\frac{-2x-y+500}{5} = 0$ Answer: $f: -0.4x - 0.2y = -100$

$\frac{1}{10}(-2x-5y+900) = 0$ $g: -0.2x - 0.5y = -90$

intersect[f, g] (200, 100)

c. Determine $D(x, y) = f_{xx} \cdot f_{yy} - \left(f_{xy}\right)^2$.

Commands: Answers:

derivative[a, x] f_{xx} $c(x,y) = -2/5$

derivative[b, y] f_{yy} $d(x,y) = -1/2$

derivative[a, y] f_{xy} $e(x,y) = -1/5$

$D(x,y) = \left(-\frac{\cancel{2}}{5}\right)\left(-\frac{1}{\cancel{2}}\right) - \left(\frac{1}{5}\right)^2 = \frac{1 \cdot 5}{5 \cdot 5} - \frac{1}{25} = \frac{4}{25} > 0$

d. Apply the second derivative test to classify each critical point found in part b.

$D(200, 100) > 0$

$f_{xx}(200, 100) = -\frac{2}{5} < 0$ Rel max at (200, 100)

e. How many of each type of item should be manufactured each week to maximize profit?

200 standard items

100 delux items

f. What is the maximum profit that is realizable in this situation?

Command: Answer:

P(200, 100)

10500

$10,500,000