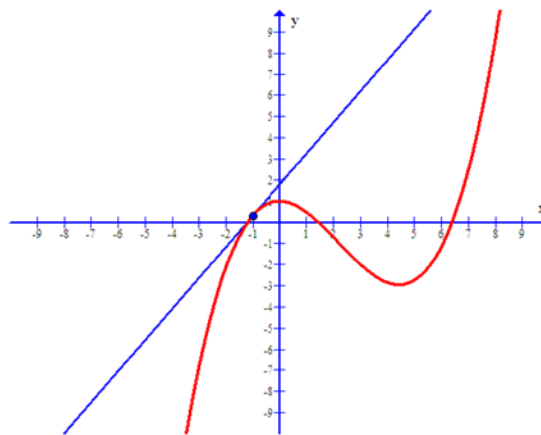**Math 1314**
Lesson 4
Limits

What is calculus?

Calculus is the study of change, particularly, how things change over time. It gives us a framework for measuring change using some fairly simple models. In this course, we will address two basic questions in this course.

1. How can we find the slope of a line that is tangent to an arbitrary curve at a given point?

2. How can we find the area of a region such as the one pictured below?
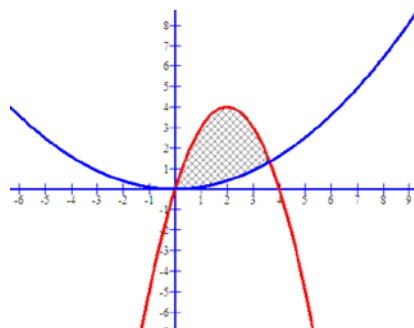
Both of these areas of inquiry rely on the concept of a *limit*. We will begin our study of calculus by looking at limits.

Limits

Finding a limit amounts to answering the following question: What is happening to the *y*-value of a function as the *x*-value approaches a specific target number? If the *y*-value is approaching a specific number, then we can state the limit of the function as *x* gets close to the target number.

Notation: "lim" is short for limit. If we write $\lim\limits_{x \to 2} f(x)$ it means, "the limit of *f(x)*

(*y* value) as the *x* values *approach* 2".

**The limit formally defined:**

We say that a function *f* has **limit** *L* as *x* approaches the target number *a*, written $\lim\limits_{x \to a} f(x) = L$

if the value *f(x)* can be made as close to the number *L* as we please by taking *x* sufficiently close to (but not equal to) *a*. Note that *L* is a single real number. Otherwise, the limit fails to exist (DNE = Does Not Exist).

Example 1: Given the graph of *f(x)* below. Find:
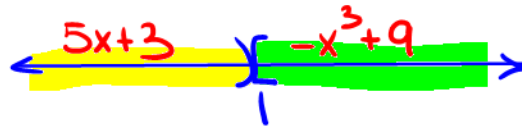
a. $\lim\limits_{x \to 0} f(x)$ $= -1$

b. $\lim\limits_{x \to 2} f(x)$ DNE

c. $\lim\limits_{x \to -2} f(x) = 3$
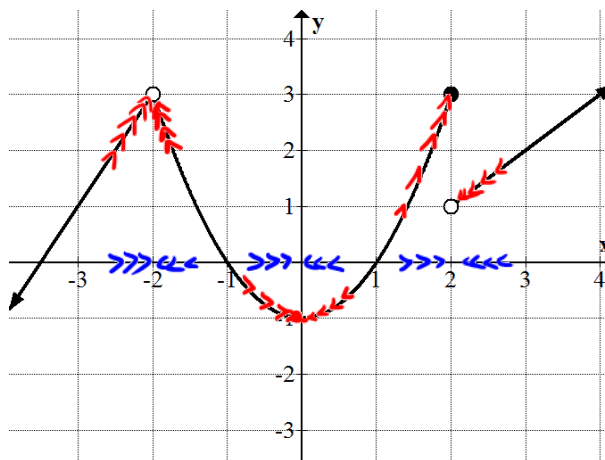
Example 2: Given the following function, what is the limit of the function as x approaches 1?

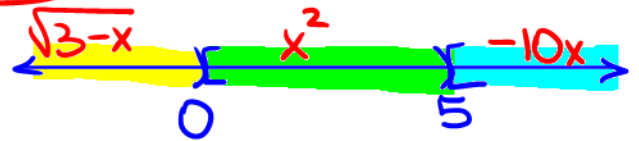$$f(x) = \begin{cases} 5x+3, & \text{if } x < 1 \\ -x^3 + 9, & \text{if } x \geq 1 \end{cases}$$

$\underline{5x+3} \qquad \underline{-x^3+9}$

$f(1) = 5(1) + 3 = \boxed{8}$

$f(1) = -(1)^3 + 9 = \boxed{8}$ } same

$\lim\limits_{x \to 1} f(x) = 8$

Example 3: Let $f(x) = \begin{cases} \sqrt{3-x}, & \text{if } x < 0 \\ x^2, & \text{if } 0 \leq x < 5 \text{, find } \lim\limits_{x \to 5} f(x). \\ -10x, & \text{if } x \geq 5 \end{cases}$

$\sqrt{3-x} \qquad x^2 \qquad -10x$
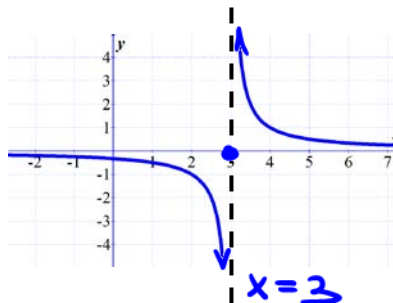
$0 \qquad 5$

$f(5) = (5)^2 = \boxed{25}$

$f(5) = -10(5) = \boxed{-50}$ } different

$\lim\limits_{x \to 5} f(x) = DNE$

As in Example 1 part b, we have seen a limit fails to exist when it approaches ==two different y-values numbers from each side of a target number== (the given x value).

A limit also fails to exist when at the target number for the x-value, the graph of the function has an asymptote. The y-values will approach $\pm\infty$.

$x=3$

$\lim\limits_{x \to 3} f(x) = DNE$

Evaluating Limits

Most often, we will not have a graph given, and will need to find a limit by algebraic methods.

Properties of limits:

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then,

1. $\lim_{x \to a}[f(x)]^r = [\lim_{x \to a} f(x)]^r = L^r$ for any real number r.

2. $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) = cL$ for any real number c.

3. $\lim_{x \to a}\left[f(x) \pm g(x)\right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$

4. $\lim_{x \to a}[f(x)g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)] = LM$.

5. $\lim_{x \to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M}$, provided $M \neq 0.$

Substitution

Example 4: Evaluate: $\lim_{x \to 2}(x^3 - 1)$

$= (2)^3 - 1 = 8 - 1 = 7$

Example 5: $\lim_{x \to 0}\dfrac{\sqrt{x + 16}}{x + 6}$

$= \dfrac{\sqrt{0 + 16}}{0 + 6} = \dfrac{\sqrt{16}}{6} = \dfrac{4}{6} = \dfrac{2}{3}$

Evaluate 6: Evaluate: $\lim_{x \to -1} 55$ $= 55$

Example 7: Evaluate: $\lim_{x \to -5}\dfrac{-x - 5}{3x}$

$= \dfrac{-(-5) - 5}{3(-5)} = \dfrac{0}{-15} = 0$

Example 8: Evaluate: $\lim_{x \to 4}\dfrac{x + 8}{x - 4}$ $= \dfrac{4 + 8}{4 - 4} = \dfrac{12}{0}$ Undefined

DNE

When substitution gives you a value in the form $\dfrac{k}{0}$, where k is any non-zero real number the limit DNE.

Lesson 4 – Limits

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Indeterminate Forms

What do you do when substitution gives you the value $\frac{0}{0}$?

This is called an ==indeterminate form==. It means that we are not done with the problem! We must try another method for evaluating the limit!!

Once we determine that a problem is in indeterminate form, we can use GGB to find the limit. Command: ==limit[<Function>,<Value>]== *This command can also be used when evaluating more complicated limits.*

Example 9: The following examples are of the indeterminate form. Use GGB to evaluate the limit. Begin by entering the function into GGB.

a. $\lim\limits_{x\to 2}\dfrac{x^2+3x-10}{x^2-4}$ $= \dfrac{(2)^2+3(2)-10}{(2)^2-4} = \dfrac{0}{0}$

Command: Answer:

$\lim\limits_{x\to 2}\dfrac{x^2+3x-10}{x^2-4} = \lim\limits_{x\to 2}\dfrac{(x+5)(\cancel{x-2})}{(\cancel{x-2})(x+2)} = \lim\limits_{x\to 2}\dfrac{x+5}{x+2} = \dfrac{7}{4}$

$\lim[f, 2]$ 1.75

b. $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}$ $= \dfrac{\sqrt{4}-2}{4-4} = \dfrac{0}{0}$

Command: Answer:

$\lim[g, 4]$ 0.25

So far we have looked at problems where the target number is a specific real number. Sometimes we are interested in finding out what happens to our function as x increases (or decreases) without bound.

Limits at Infinity

Consider the function $f(x) = \dfrac{2x^2}{x^2+1}$. As the value of x get larger and larger, f(x)

approaches 2. We can see this by looking at the table below or its graph.

x	10	1000	100,000	1,000,000
$f(x)$	1.98…	1.99…	1.999…	1.999…
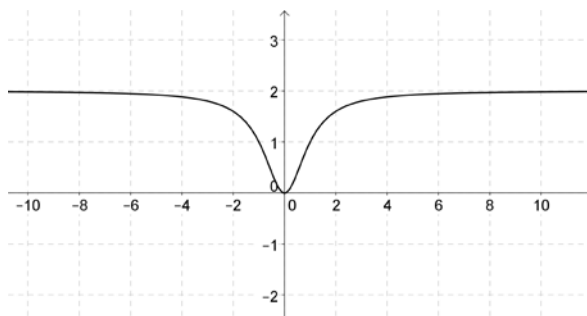
If a limit at infinity exists and it's equal to a single real number L then they are written as $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to -\infty} f(x) = L$.

Limits at infinity problems often involve rational expressions (fractions). The technique we can use to evaluate limits at infinity is to recall some rules from Algebra used to find horizontal asymptotes. These rules came from "limits at infinity" so they'll surely work for us here.

The highest power of the variable in a polynomial is called the **degree** of the polynomial.

We can compare the degree of the numerator with the degree of the denominator and come up with some generalizations.

deg (N) < deg (D)

- If the degree of the numerator is smaller than the degree of the denominator, then

  $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 0.$

  deg (N) = deg (D)

- If the degree of the numerator is the same as the degree of the denominator, then you can

  find $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$ by making a fraction from the leading coefficients of the numerator and

  denominator and then reducing to lowest terms. deg (N) > deg (D)

- If the degree of the numerator is larger than the degree of the denominator, then it's best to use the limit command in GGB.

Example 10: Evaluate: $\lim\limits_{x\to\infty}\dfrac{5x^2+3x+4}{4x^2-2x+8}$ $=\dfrac{5}{4}$

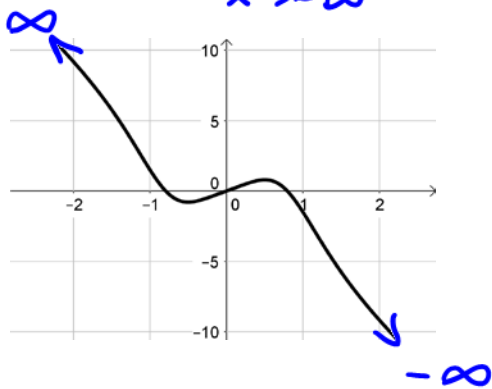Example 11: Evaluate: $\lim\limits_{x\to-\infty}\dfrac{4x+5}{x^2-9x+9}$ $=0$

Example 12: Evaluate: $\lim\limits_{x\to\infty}\dfrac{-5x^5+2x}{x^4+1}$ DNE   or $-\infty$

Command: $\lim\limits_{x\to-\infty} f(x)=\infty$ Answer:

∞

$-\infty$

Example 13: $\lim\limits_{x\to-\infty}\dfrac{2x^3-3x+9}{x}$ $=\infty$

Command: Answer:

$g(x)=(2x^3-3x+9)/x$

$-\infty$

Applications Involving Limits

Example 14: The average cost in dollars of constructing each skateboard when x skateboards are produced can be modeled by the function $\overline{C}(x) = 12.5 + \dfrac{123,500}{x}$. What is the average cost per skateboard if the number of boards produced gets larger?

$$\lim_{x \to \infty} \left(12.5 + \frac{123\,500}{x}\right) = \lim_{x \to \infty}(12.5) + \lim_{x \to \infty} \frac{123500}{x}$$

$$= 12.5 + 0 = 12.5$$

Example 15: The concentration of a drug in a patient's bloodstream t hours after the drug is administered can be modeled by the function $f(t) = \dfrac{0.2t^2}{100t^2 + t + 1}$, where $f(t)$ is given in mg/cc.

Find the amount of the drug that is in the patient's bloodstream as the amount of time since it was administered increases.

$$t \to \infty$$

$$\lim_{t \to \infty} \frac{0.2t^2}{100t^2 + t + 1} = \frac{0.2}{100} = 0.002 \text{ mg/cc}$$