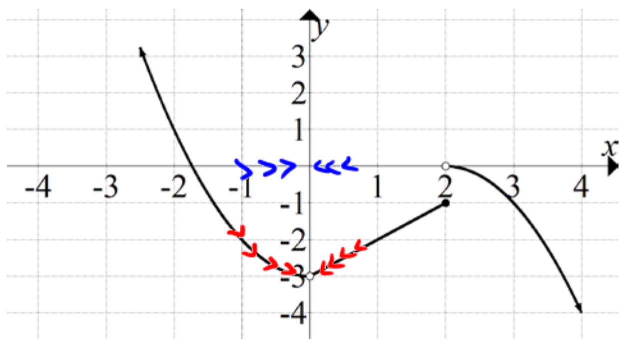
Math 1314
Lesson 5
One-Sided Limits and Continuity

Sometimes we are only interested in the behavior of a function when we look from one side and not from the other. In this case, we are looking at a **one-sided limit**.

We write $\lim_{x \to a^+} f(x)$ for a **right-hand limit**. We write $\lim_{x \to a^-} f(x)$ for a **left-hand limit**.

Theorem: Let *f* be a function that is defined for all values of *x* close to the target number *a*, except perhaps at *a* itself. Then $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.

Example 1: Given the graph of f below:



Find each of the following limits, if it exist.

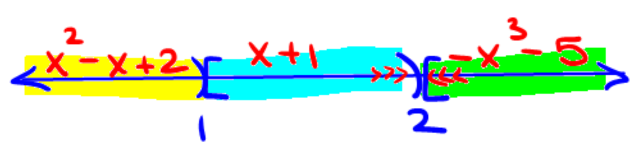
a. $\lim_{x \to 0^-} f(x)$ $= -3$

b. $\lim_{x \to 0^+} f(x)$ $= -3$

c. $\lim_{x \to 0} f(x)$ $= -3$

same

Example 2: Suppose $f(x) = \begin{cases} x^2 - x + 2, & x < 1 \\ x + 1, & 1 \le x < 2 \\ -x^3 - 5, & x \ge 2 \end{cases}$. Find each of the following limits, if it exist.



a. $\lim_{x \to 2^-} f(x)$

$= 2 + 1 = 3$

b. $\lim_{x \to 2^+} f(x)$

$= -(2)^3 - 5 = -13$

c. $\lim_{x \to 2} f(x)$

DNE

$\neq$

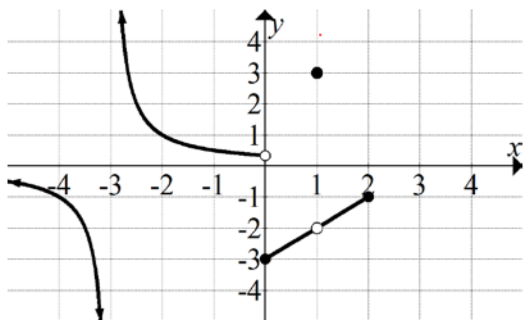
Continuity at a Point

A function is a **continuous** at a point if its graph has no gaps, holes, breaks or jumps at that point. Stated a bit more formally, a function f is said to be continuous at the point $x = a$ if the following three conditions are met:

1. $f(a)$ is defined 2. $\lim_{x \to a} f(x)$ exists 3. $\lim_{x \to a} f(x) = f(a)$

If a function is not continuous at $x = a$, then we say it is discontinuous there.

Example 3: The graph of a function given below is discontinuous at some values of x. State the x-values of where the function is discontinuous then state why the function is discontinuous at each one of those points.



a. Discontinuous at: $x = -3$ NO! Inf. discon.
- Is $f(-3)$ defined?
- Does $\lim_{x \to} f(x)$ exists?
- Does $\lim_{x \to} f(x) = f(\quad)$?

b. Discontinuous at: $x = 0$
- Is $f(0)$ defined? YES $f(0) = -3$
- Does $\lim_{x \to 0} f(x)$ exists? NO! Jump.
- Does $\lim_{x \to} f(x) = f(\quad)$?

c. Discontinuous at: $x = 1$
- Is $f(1)$ defined? YES $f(1) = 3$ Removable
- Does $\lim_{x \to 1} f(x)$ exists? YES $\lim_{x \to 1} f(x) = -2$
- Does $\lim_{x \to 1} f(x) = f(1)$? NO!

$-2 \neq 3$

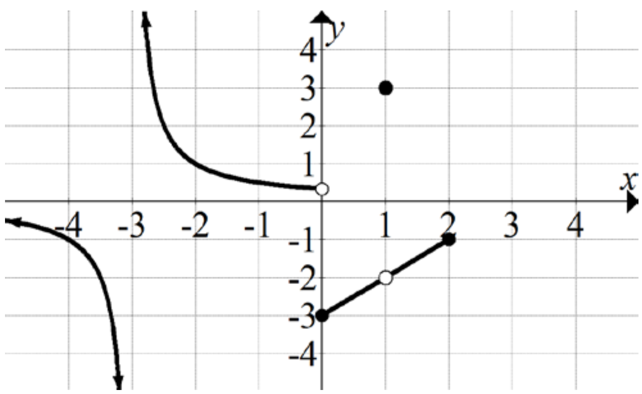
Discontinuities

A function can have a removable discontinuity, a jump discontinuity or an infinite discontinuity.

Let $f(x)$ be discontinuous at $x = a$. Then:

If	Type of Discontinuity
$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$	Jump
$\lim_{x \to a} f(x) \neq f(a)$	Removable
$\lim_{x \to a^-} f(x) \to \pm\infty$ or $\lim_{x \to a^+} f(x) \to \pm\infty$	Infinite

Example 4: Let's revisit the graph from Example 3.



State the type of discontinuity at:

a. x = -3

Inf.

b. x = 0

Jump

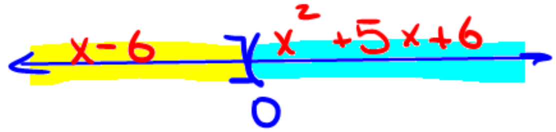
c. x = 1

Removable

Example 5: Let $f(x)=\begin{cases} x-6, & x\le 0 \\ x^2+5x+6, & x>0 \end{cases}$ is the function continuous at $x = 0$?

We need to check:

1. Is $f(0)$ defined? YES!



$f(0) = 0-6 = -6$

2. Does $\lim_{x\to 0} f(x)$ exist? Must check $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$. NO!

$\lim_{x\to 0^-} f(x) = 0-6 = \boxed{-6}$ $\neq$

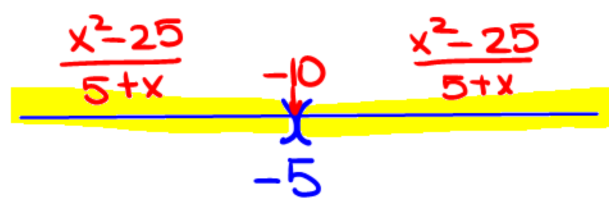
$\lim_{x\to 0^+} f(x) = 0^2+5(0)+6 = \boxed{6}$

~~3.~~ $\lim_{x\to 0} f(x) = f(0)$? *i.e. Compare #1 and #2 above.*

No need to check! Jump

$f(x)$ is discontinuous at $x=0$

Example 5: Let $f(x)=\begin{cases} \dfrac{x^2-25}{5+x}, & x\ne -5 \\ -10, & x=-5 \end{cases}$ is the function continuous at $x=-5$?



We need to check:

1. Is $f(-5)$ defined? YES

$f(-5) = \boxed{-10}$

2. Does $\lim_{x\to -5} f(x)$ exist? Must check $\lim_{x\to -5^-} f(x)$ and $\lim_{x\to -5^+} f(x)$. YES

$$\lim_{x\to -5} f(x) = \frac{(-5)^2-25}{5+(-5)} = \frac{0}{0}$$

$$\lim_{x\to -5} \frac{x^2-25}{5+x} = \lim_{x\to -5} \frac{(x-5)\cancel{(x+5)}}{\cancel{(5+x)}} = -5-5 = \boxed{-10}$$

3. $\lim_{x\to -5} f(x) = f(-5)$? *i.e. Compare #1 and #2 above.* YES

f is continuous at $x=-5$