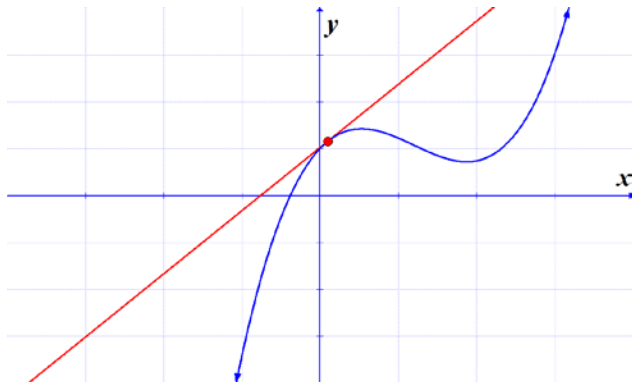
Math 1314
Lesson 6
The Limit Definition of the Derivative; Rules for Finding Derivatives

We now address the first of the two questions of calculus, the tangent line question.

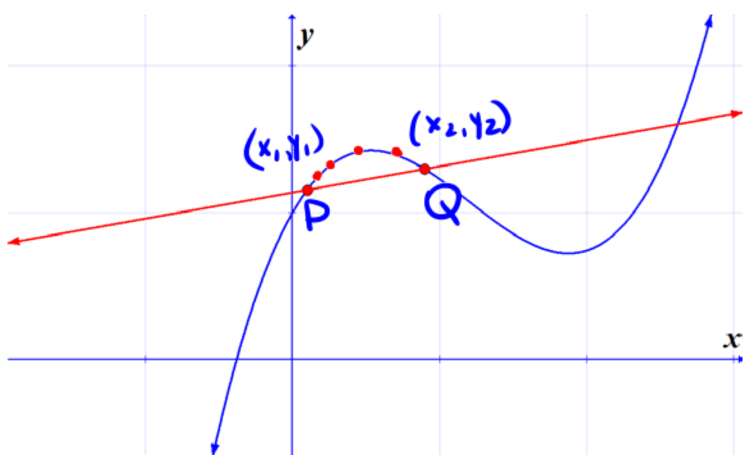
We are interested in finding the slope of the tangent line at a specific point.



We need a way to find the slope of the tangent line analytically for every problem that will be exact every time.

We can draw a secant line across the curve, then take the coordinates of the two points on the curve, *P* and *Q*, and use the slope formula to approximate the slope of the tangent line.

$m = \frac{y_2 - y_1}{x_2 - x}$

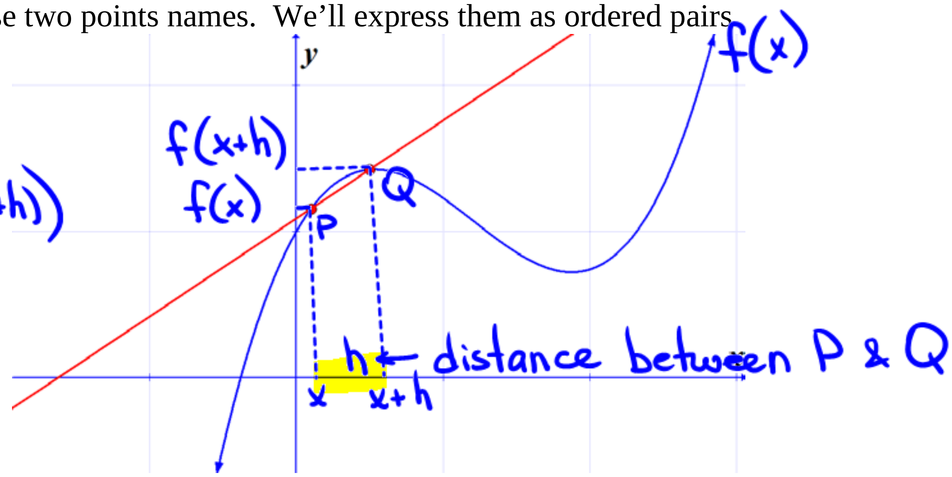


Now suppose we move point *Q* closer to point *P*. When we do this, we'll get a better approximation of the slope of the tangent line. When we continue to move point *Q* even closer to point *P*, we get an even better approximation. We are letting the distance between *P* and *Q* get smaller and smaller.

Now let's give these two points names. We'll express them as ordered pairs.

$P(\overset{x_1}{x}, \overset{y_1}{f(x)})$

$Q(\overset{x_2}{x+h}, \overset{y_2}{f(x+h)})$



Now we'll apply the slope formula to these two points.

$$m=\frac{\overbrace{f(x+h)-f(x)}^{y_2-y_1}}{\underbrace{(\cancel{x}+h)}_{x_2}-\underbrace{\cancel{x}}_{x_1}}=\frac{f(x+h)-f(x)}{h}$$

This expression is called a **difference quotient** also called the **average rate of change.**

The last thing that we want to do is to let the distance between P and Q get arbitrarily small, so we'll take a limit.

This gives us the definition of the **slope of the tangent line**.

The slope of the tangent line to the graph of f at the point $P(x, f(x))$ is given by

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

provided the limit exists.

We find the **instantaneous rate of change** when we take the limit of the difference quotient.

The **derivative of *f* with respect to *x*** is the function f' (read "f prime") defined by

$f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$. The domain of $f'(x)$ is the set of all x for which the limit exists.

We can use the derivative of a function to solve many types of problems. But first we need a method for finding the derivative.

Now that we know what the derivative is, we need to be able to find the derivative of a function. We can use an algebraic process (limit definition of the derivative) to find the derivative but it isn't always convenient. Fortunately, there are some rules for finding derivatives which will make this easier.

Rules for Finding Derivatives

We can use the limit definition of the derivative to find the derivative of every function, but it isn't always convenient. Fortunately, there are some rules for finding derivatives which will make this easier.

First, a bit of notation: $\frac{d}{dx}[f(x)]$ is a notation that means "the derivative of f with respect to x, evaluated at x."

Rule 1: The Derivative of a Constant

$\boxed{\frac{d}{dx}[c]=0,}$ where c is a constant.

Example 1: Find the derivative of each function.

a. $f(x)=-17$

$f'(x)=0$

b. $g(x)=\sqrt{11}$

$g'(x)=0$

Rule 2: The Power Rule

$\boxed{\frac{d}{dx}[x^n]=nx^{n-1}}$ for any real number n

Example 2: Find the derivative of each function.

a. $f(x)=x^5$

$f'(x)=5x^{5-1}$
$=5x^4$

b. $g(x)=x^{-10}$

$g'(x)=-10x^{-10-1}$
$=-10x^{-11}$

c. $h(x)=\frac{1}{x^3}=x^{-3}$

$h'(x)=-3x^{-3-1}$
$=-3x^{-4}=\frac{-3}{x^4}$

d. $j(x)=\sqrt{x}=x^{1/2}$

$j'(x)=\frac{1}{2}x^{\frac{1}{2}-1}$
$=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2}\cdot\frac{1}{x^{1/2}}$
$=\frac{1\cdot\sqrt{x}}{2\sqrt{x}\cdot\sqrt{x}}=\frac{\sqrt{x}}{2x}$

Rule 3: Derivative of a Constant Multiple of a Function

$\boxed{\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]}$ where c is any real number

Example 3: Find the derivative of each function.

a. $f(x) = -6x^4$

$f'(x) = -6(x^4)'$
$= -6 \cdot 4x^{4-1} = \boxed{-24x^3}$

b. $g(x) = \frac{1}{4}x^{-4}$

$g'(x) = \cancel{\frac{1}{4}}(-\cancel{4}x^{-4-1})$
$= -x^{-5} = \boxed{-\frac{1}{x^5}}$

c. $h(x) = 5x$

$h'(x) = 5(x^{1-1}) = 5\cancel{x^0}$ 1
$= 5$

Rule 4: The Sum/Difference Rule

$\boxed{\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]}$

Example 4: Find the derivative: $f(x) = 10x^4 + 3x^2 - 6x + 5$.

$f'(x) = 10(4x^{4-1}) + 3(2x^{2-1}) - 6 + 0$
$= 40x^3 + 6x - 6$

$\sqrt[n]{x} = x^{\frac{1}{n}}$

$x^{\frac{m}{n}} = \sqrt[n]{x^m}$

$x^{\frac{1}{6}}$ x^{-4}

Example 5: Find the derivative: $f(x) = -2x^7 - 6\sqrt[6]{x} + \frac{1}{2x^4} - 1$.

$f'(x) = -2(7x^{7-1}) - \cancel{6}(\frac{1}{\cancel{6}}x^{\frac{1}{6}-1}) + \frac{1}{\cancel{2}}(-\cancel{4}x^{-4-1})$ 2

$= -14x^6 - x^{-\frac{5}{6}} - 2x^{-5} = -14x^6 - \frac{1}{\sqrt[6]{x^5}} - \frac{2}{x^5}$

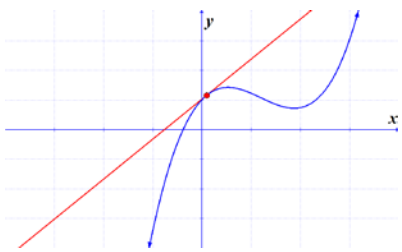
Note, there are many other rules for finding derivatives "by hand." We will not be using those in this course. Instead, we will use GeoGebra for finding more complicated derivatives.

Let's do a few applications.

A common use of rate of change is to describe the motion of an object. The function gives the position of the object with respect to time, so it is usually a function of *t* instead of *x*. If the object changes position over time, we can compute its rate of change, which we refer to as **velocity**. We can find either the average rate of change or the instantaneous rate of change, depending on the question posed.

Velocity can be positive, negative or zero. If you throw a rock up in the air, its velocity will be positive while it is moving upward and will be negative while it is moving downward. We refer to the absolute value of velocity as **speed**. Velocity has two components: speed and direction.

- *rate of change, velocity, instantaneous velocity => the derivative*



- *average rate of change, difference quotient => an average*
  An interval MUST be given!!

Example 6: Suppose the distance covered by a car can be measured by the function $f(t) = 4t^2 + 32t$, where $f(t)$ is given in feet and *t* is measured in seconds.

a. Find the rate of change of the car when $t = 4$.

$f'(t) = 4(2t) + 32 = 8t + 32$

$f'(4) = 8(4) + 32 = 64$ ft/sec.

b. Find the average velocity of the car over the interval [1, 6].

$t \quad t+h$

$h = 6 - 1 = 5$

Recall: $\dfrac{f(t+h) - f(t)}{h}$.

$$= \frac{f(6) - f(1)}{5} = \frac{\overbrace{[4(6)^2 + 32(6)]}^{f(6)} - \overbrace{[4(1)^2 + 32(1)]}^{f(1)}}{5} = 60 \text{ ft/sec}$$

Example 7: The median price of a home in one part of the US can be modeled by the function $P(t) = -0.01363t^2 + 9.2637t + 125.84$, where $P(t)$ is given in thousands of dollars and t is the number of years since the beginning of 1995. According to the model, at what rate were median home prices changing at the beginning of 2005?

1995 → 0

2005 → 10

$$P'(t) = -0.01363(2t) + 9.2637$$
$$= -0.02726t + 9.2637$$

$$P'(10) = -0.02726(10) + 9.2637 = \$8.9911 \text{ thousand}$$

Example 8: A country's gross domestic product (in millions of dollars) is modeled by the function $G(t) = -2t^3 + 45t^2 + 20t + 6000$ where $0 \le t \le 11$ and $t = 0$ corresponds to the beginning of 1997. What was the average rate of growth of the GDP over the period 1999 – 2004?

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$h = 7 - 2 = 5$

$$\frac{G(t+h) - G(t)}{h}$$

$$= \frac{G(7) - G(2)}{5} = \$291 \text{ million/year}$$