## MATH 2312    Algebra Review Part 1

### Factoring and square root

$$a^2 - b^2 = (a-b)(a+b)$$
$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$\cancel{a^3 + b^3 = (a-b)(a^2 - ab + b^2)}$$
$$\sqrt{a^2} = |a|$$

$$\sqrt{(-3)^2} = |-3| = 3$$

**Example:** Simplify the following expression:

$$\frac{(x^3+27)\sqrt{(x-7)^2}}{(x^2-3x+9)} = \frac{(x+3)\cancel{(x^2-3x+9)}|x-7|}{\cancel{(x^2-3x+9)}}$$

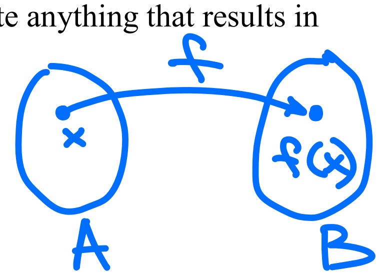
$$= (x+3)|x-7|$$

### Functions and Domain

A function $f$ is a rule that assigns to each element $x$ in a set $A$ exactly one element $f(x)$ in a set $B$.

The set A is called the domain and is the set of all valid inputs for the function.

To determine the domain, start with all real numbers and then eliminate anything that results in zero denominator or even roots of negative numbers.




**Example:** Find the domain for each of the following functions.

a. $f(x) = 3x^2 - 5x + 2$   Poly

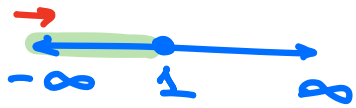
$(-\infty, \infty)$ or $\mathbb{R}$

b. $f(x) = \sqrt{1-x}$

$1 - x \geq 0$

$1 \geq x$

$x \leq 1$




D: $(-\infty, 1]$

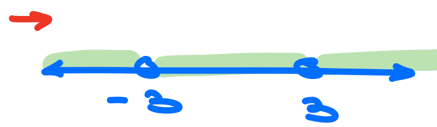
c. $f(x) = \frac{2x+6}{x^2-9} + \frac{1}{x^2+1}$ ← no problem

$x^2 + 1 \neq 0$

$x^2 - 9 \neq 0$

$\sqrt{x^2} \neq \sqrt{9}$

$x \neq \pm 3$




D: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

$5 - \cos(2x) \geq 0$

$5 \geq \cos(2x)$

d. $f(x) = \sqrt{5 - \cos 2x}$

$\cos(2x) \leq 5$ Always true

D: $\mathbb{R}$

$-1 \leq \cos(\ ) \leq 1$

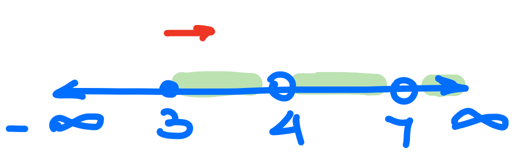
e. $f(x) = \frac{\sqrt{x-3}}{x^2-11x+28}$

$$f(x) = \frac{\sqrt{x-3}}{(x-4)(x-7)}$$

$x - 3 \geq 0$ AND $x - 4 \neq 0$ $x - 7 \neq 0$

$x \geq 3$ $x \neq 4$ $x \neq 7$

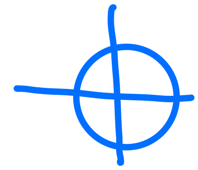





$[3, 4) \cup (4, 7) \cup (7, \infty)$

f. $f(x) = \frac{x-2}{\sin x - 1}$

$\sin x - 1 \neq 0$



$\sin x \neq 1$

$x \neq \frac{\pi}{2} + 2\pi k \quad k = \text{integer}$

$\mathbb{R} - \left\{\frac{\pi}{2} + 2\pi k,\ k = \text{integer}\right\}$

$\ldots \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{5\pi}{2}\right) \cup \left(\frac{5\pi}{2}, \frac{9\pi}{2}\right) \cup \ldots,$

**Difference Quotient**

**Definition:** Let $f(x)$ be a function. The difference quotient is the expression

$$\boxed{\frac{f(x+h)-f(x)}{h}}$$

**Example:** Find the difference quotient for $f(x) = x^2 - 3x + 1$

**Step 1:** Evaluate $f(x+h)$

$$f(x+h) = (x+h)^2 - 3(x+h) + 1$$

$$= x^2 + 2xh + h^2 - 3x - 3h + 1$$

**Step 2:** Evaluate $f(x+h) - f(x)$

$$f(x+h) - f(x) =$$

$$x^2 + 2xh + h^2 - 3x - 3h + 1 - (x^2 - 3x + 1)$$

$$\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h + \cancel{1} - \cancel{x^2} + \cancel{3x} - \cancel{1}$$

$$= 2xh + h^2 - 3h$$

**Step 3:** Divide $f(x+h) - f(x)$ by $h$

$$\frac{2xh + h^2 - 3h}{h} = \boxed{2x + h - 3}$$

**Example:** Find the difference quotient for $f(x) = \frac{1}{x-3}$

**Step 1:** Evaluate $f(x+h)$

$$f(x+h) = \frac{1}{x+h-3}$$

**Step 2:** Evaluate $f(x+h) - f(x)$

$$\frac{1}{x+h-3}\cdot\frac{x-3}{x-3} - \frac{1}{x-3}\cdot\frac{x+h-3}{x+h-3} = \frac{x-3-(x+h-3)}{(x+h-3)(x-3)}$$

$$= \frac{\cancel{x}\cancel{-3}\cancel{-x}-h\cancel{+3}}{(x+h-3)(x-3)} = \frac{-h}{(x+h-3)(x-3)}$$

**Step 3:** Divide $f(x+h) - f(x)$ by $h$ (Multiply by $\frac{1}{h}$)

$$\frac{-\cancel{h}}{(x+h-3)(x-3)} \cdot \frac{1}{\cancel{h}} = \boxed{\frac{-1}{(x+h-3)(x-3)}}$$

**Composition of Functions**

**Definition:** The composition of functions $f$ and $g$ is defined as $(f \circ g)(x) = f(g(x))$.

The domain of $f \circ g$ is the set of all $x$ such that $x$ is in the domain of $g(x)$ and $g(x)$ is in the domain of $f(x)$.

Note:

- $(fg)(x)$ and $f(g(x))$ have different meaning!!! Be careful about the notation.
- In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

**Example:** Let $f(x) = 2x - 3$ and $g(x) = x^2 + 2$. Find

a. $(f \circ g)(x)$ $= f(g(x))$

$= 2(x^2+2) - 3 = 2x^2 + 4 - 3$

$= \boxed{2x^2+1}$

NOT the same

b. $(g \circ f)(x)$ $= g(f(x))$

$= (2x-3)^2 + 2$

$= 4x^2 - 12x + 9 + 2 = \boxed{4x^2 - 12x + 11}$

$-2(2x)(3)$

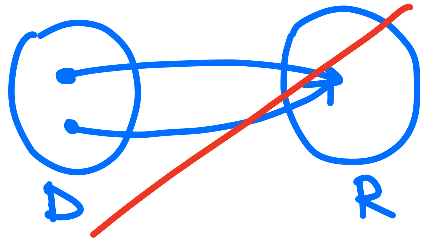
**Example:** Let $f(x) = 5x^2 - 1$ and $g(x) = 3\cos x - 1$. Evaluate $(f \circ g)(0)$

$(f \circ g)(0) = f(g(0))$ INSIDE → OUT

1. $g(0) = 3\cos(0) - 1 = 3(1) - 1 = 2$

2. $f(2) = 5(2)^2 - 1 = 20 - 1 = 19$

$(f \circ g)(0) = \boxed{19}$




**Inverse Functions**

A function is **one-to-one** (1-1) if no two elements in the domain have the same image. If a function is 1-1 then it has an inverse.

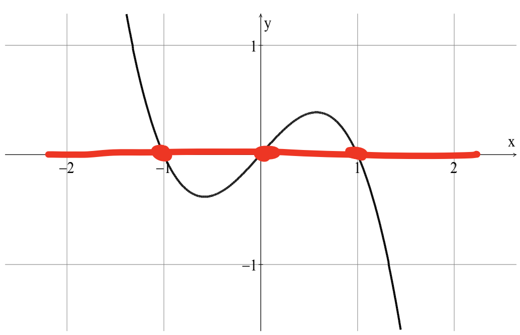
The inverse of a function $f$ is denoted by $f^{-1}$. Note that $f^{-1}(x) \neq \frac{1}{f(x)}$.

Given the graph of a function, we can determine if that function has an inverse function by applying the Horizontal Line Test.

**The Horizontal Line Test**
A function $f$ has an inverse function, $f^{-1}$, if there is no horizontal line that intersects the graph in more than one point.

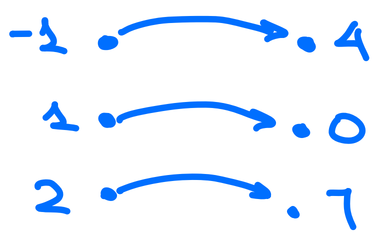
**Example:** Which of the following are one-to-one functions?

a.



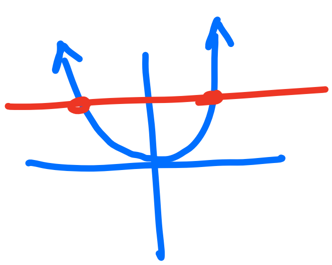

NO

b. {(-1, 4), (1, 0), (2, 7)}




YES

c. $f(x) = x^2$



$x = 1 \quad f(1) = 1$

$x = -1 \quad f(-1) = 1$

NO

**Domain and Range**

The inverse function reverses whatever the first function did; therefore, the domain of $f$ is the range of $f^{-1}$ and the range of $f$ is the domain of $f^{-1}$.

**Example:** Suppose that $f$ and $g$ are 1-1 functions and that $f(3) = 7$, $f(7) = 2$, $g(7) = 4$, and $g(3) = 0$. Find $f^{-1}(g^{-1}(4))$.

$3 \xrightarrow{f} 7$, $7 \xrightarrow{f^{-1}} 3$

$7 \xrightarrow{f} 2$, $2 \xrightarrow{f^{-1}} 7$

$7 \xrightarrow{g} 4$, $4 \xrightarrow{g^{-1}} 7$

$3 \xrightarrow{g} 0$, $0 \xrightarrow{g^{-1}} 3$

$f^{-1}(g^{-1}(4))$

$= f^{-1}(7) = 3$

**Property of Inverse Functions**

Let $f$ and $g$ be two functions such that $(f \circ g)(x) = x$ for every $x$ in the domain of $g$ and $(g \circ f)(x) = x$ for every $x$ in the domain of $f$ then ***f* and *g* are inverses of each other.**

Finding the inverse of a one-to-one function:

1. Replace $f(x)$ by $y$.
2. Exchange $x$ and $y$.
3. Solve for $y$.
4. Replace $y$ by $f^{-1}$.
5. Verify (i.e. check $(f \circ f^{-1})(x) = x$ AND $(f^{-1} \circ f)(x) = x$)

**Example:** Let $f(x) = 2x^3 + 3x + 5$. Evaluate $f^{-1}(5)$ and $f^{-1}(10)$.

$f^{-1}(5) = a$

$f(a) = 5$

$2a^3 + 3a + 5 = 5$

$\quad -5 \quad -5$

$2a^3 + 3a = 0$

$a(2a^2 + 3) = 0$

$a = 0 \qquad 2a^2 + 3 \neq 0$

$f(0) = 5$

$\boxed{f^{-1}(5) = 0}$

$f^{-1}(10) = b$

$f(b) = 10$

$2b^3 + 3b + 5 = 10$

Guess: $b = 1$

$2(1)^3 + 3(1) + 5 = 10$ ✓

$f(1) = 10$

$\boxed{f^{-1}(10) = 1}$

**Example:** Let $f(x) = 2x^3 + 12$ and $g(x) = 4\sin\left(\frac{\pi}{6}x\right)$. Evaluate $(g \circ f^{-1})(10)$.

$= g(f^{-1}(10)) = g(-1)$

$f^{-1}(10) = a$

$f(a) = 10$

$2a^3 + 12 = 10$

$2a^3 = -2$

$\sqrt[3]{a^3} = \sqrt[3]{-1}$

$a = -1$

$f^{-1}(10) = -1$

$= 4\sin\left(\frac{\pi}{6}(-1)\right)$

$= 4\sin\left(-\frac{\pi}{6}\right)$

$= -4\sin\left(\frac{\pi}{6}\right)$

$= -4 \cdot \left(\frac{1}{2}\right) = \boxed{-2}$