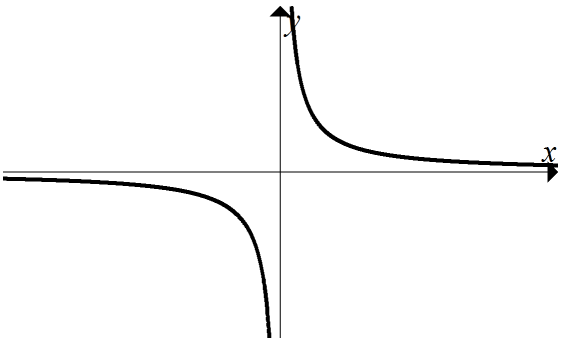
MATH 2312 Algebra Review Part 2

Asymptotes of Functions

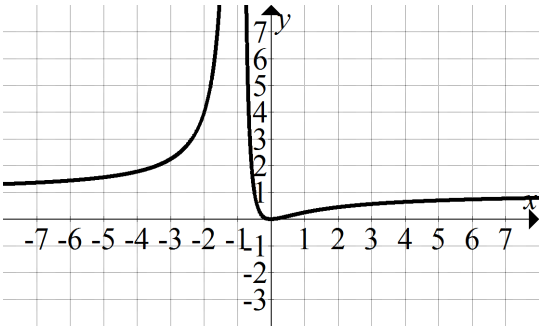
The line $x = a$ is a **vertical asymptote** of the graph of a function f if $f(x)$ increases or decreases without bound as x approaches a **.**

Examples:

Given $f(x) = \frac{1}{x}$, the line $x = 0$ (y-axis) is its vertical asymptote.



Given $f(x) = \frac{x^2}{(x+1)^2}$, the line $x = -1$ is its vertical asymptote.



Holes: The graph of the function has a hole if there is a common factor of the numerator and denominator.

Finding Vertical Asymptotes and Holes Algebraically

1. Factor the numerator and denominator as much as possible.
2. Look at each factor in the denominator.
    - If a factor cancels with a factor in the numerator, then there is a hole where that factor equals zero.
    - If a factor does not cancel, then there is a vertical asymptote where that factor equals zero.

Example: Find the vertical asymptotes for $f(x) = \frac{5x+2}{(x+2)(x-3)}$.

Example: Find the vertical asymptotes for $f(x) = \frac{x+5}{x^3-25x}$.

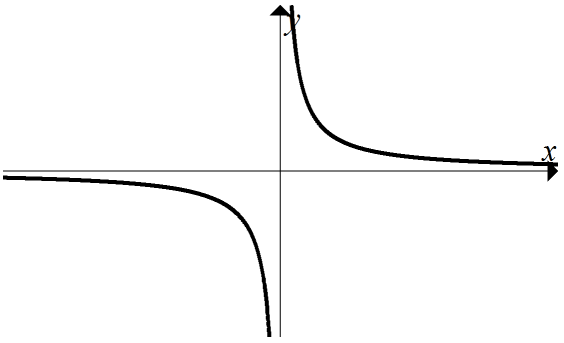
Example: Find all vertical asymptotes for $f(x) = \frac{2x}{1+2\sin x}$ in the interval $[0, 2\pi)$.

Horizontal Asymptotes

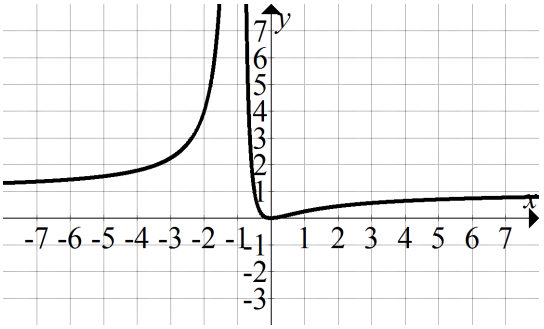
The line $y = b$ is a **horizontal asymptote** of the graph of a function f if $f(x)$ approaches b as x increases (goes to infinity) or decreases (goes to negative infinity) without bound.

Examples:

Given $f(x) = \frac{1}{x}$, the line $y = 0$ (x-axis) is its horizontal asymptote.



Given $f(x) = \frac{x^2}{(x+1)^2}$, the line $y = 1$ is its horizontal asymptote.



Horizontal asymptotes really have to do with what happens to the *y*-values as *x* becomes very large or very small. If the *y*-values approach a particular number at the far left and far right ends of the graph, then the function has a horizontal asymptote.

Finding Horizontal Asymptotes of Rational Functions

Let $f(x) = \frac{P(x)}{Q(x)}$. Shorthand: degree of $f = deg(f)$, numerator = N, denominator = D

1. If $deg(N) > deg(D)$ then there is no horizontal asymptote.
2. If $deg(N) < deg(D)$ then there is a horizontal asymptote and it is $y = 0$ (*x*-axis).
3. If $deg(N) = deg(D)$ then there is a horizontal asymptote and it is $y = \frac{a}{b}$, where

   a is the leading coefficient of the numerator.

   b is the leading coefficient of the denominator.

Note: A rational function may have several vertical asymptotes, but only at most one horizontal asymptote. In addition, a graph cannot cross a vertical asymptote, but may cross a horizontal asymptote.

Example: Find the horizontal asymptote, if there is one, of:

a. $f(x) = \frac{x^9+3x^2}{x^6+4x^5-5}$ $deg(N)$ _____ $deg(D)$; therefore, _____

b. $f(x) = \frac{x^2+2x+3}{2x^2+6x-1}$ $deg(N)$ _____ $deg(D)$; therefore, _____

c. $f(x) = \frac{1}{-x^3+5x}$ $deg(N)$ _____ $deg(D)$; therefore, _____

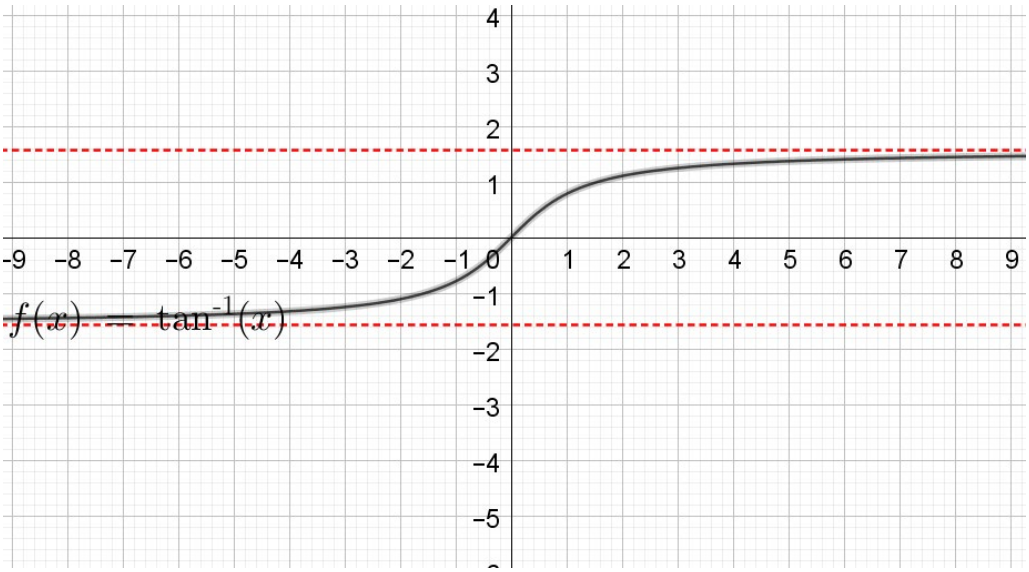
For those examples above that do have a horizontal asymptote, determine whether the graph of the function crosses it.

We have discussed horizontal asymptotes of rational functions but many other types of functions have horizontal asymptotes. For example, inverse trigonometric functions, exponential functions.

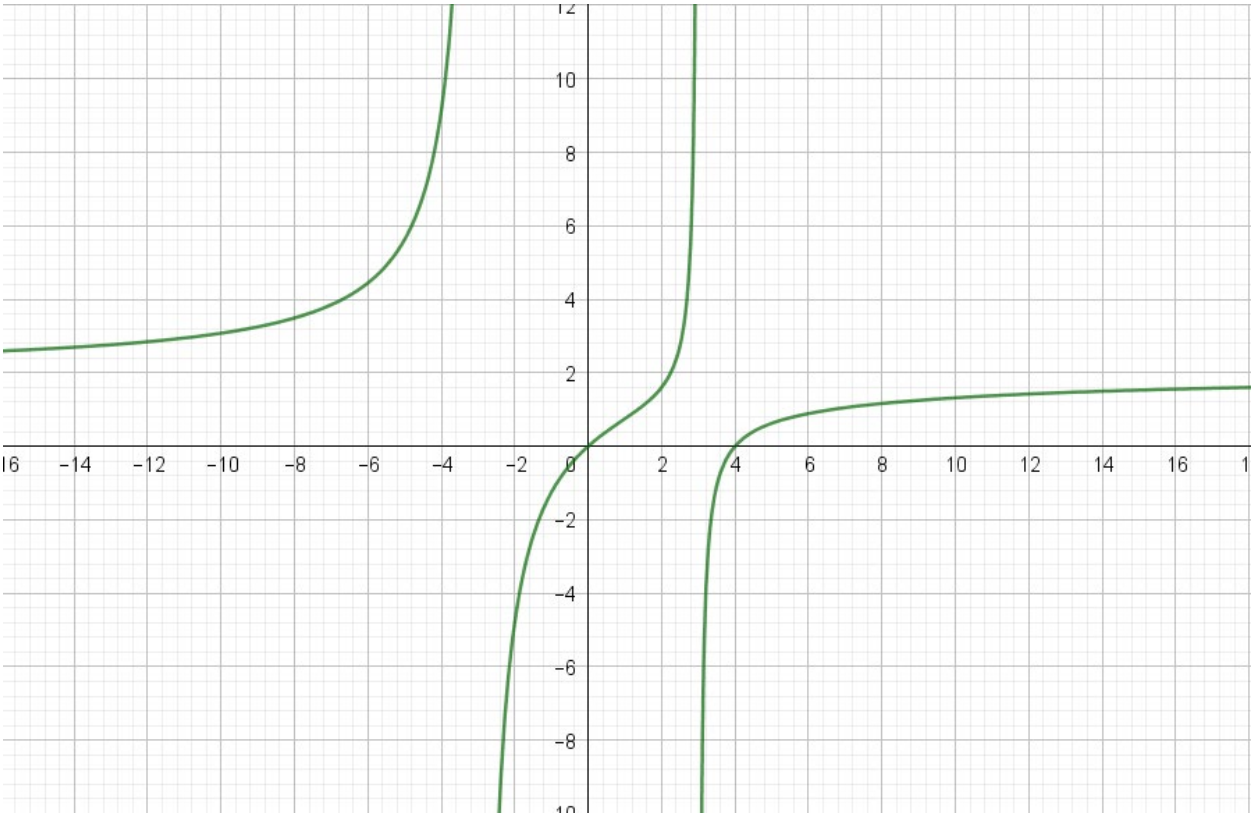
Note: Polynomial functions do NOT have asymptotes (neither vertical nor horizontal).

Note: A function may have at most two horizontal asymptotes.

Example:



Example: Given the graph below, which of the following can be this function?



A. $f(x) = \frac{x^2-4x}{x^2-9}$

B. $f(x) = \frac{2x^2+8x}{x^2-9}$

C. $f(x) = \frac{2x^2-8x}{x^2+9}$

D. $f(x) = \frac{x^2-4x}{2x^2-18}$

E. $f(x) = \frac{x^2-8}{x^2-9}$

F. $f(x) = \frac{2x^2-8x}{x^2-9}$