Math 1330
Algebra Review Part 3

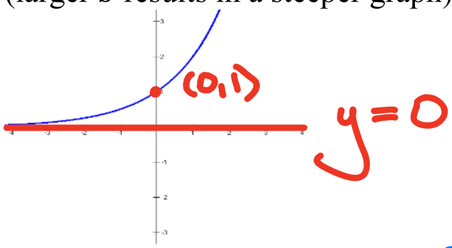
Exponential Functions

Functions whose equations contain a variable in the exponent are called **exponential functions.**

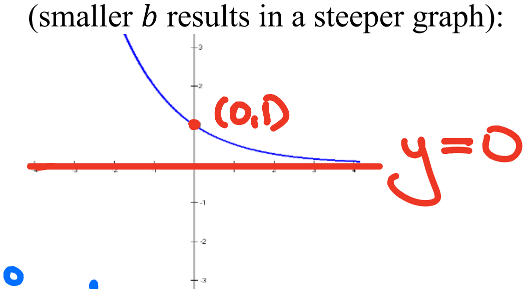
The **exponential function f with base b** is defined by $f(x) = b^x$ ($b > 0$ *and* $b \neq 1$) and x is any real number.

If $b = e$ (*the natural base*, $e \approx 2.7183$), then we have $f(x) = e^x$.

If $b > 1$, the graph of $f(x) = b^x$ looks like
(larger b results in a steeper graph):



If $0 < b < 1$, the graph of $f(x) = b^x$ looks like
(smaller b results in a steeper graph):



$f(0) = b^0 = 1$
$b \neq 0$

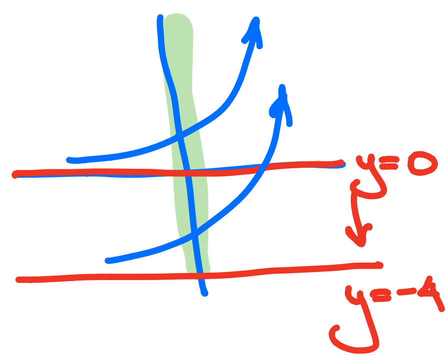
Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Key Point: (0,1)
Both graphs have a horizontal asymptote of y = 0 (the x-axis).

Example: Find the domain, range, and horizontal asymptote of the following functions.

a. $f(x) = 3^{x+1} - 4$

Basic: $f(x) = 3^x$
1 left
4 down

Domain: $\mathbb{R}$ or $(-\infty, \infty)$
Range: $(-4, \infty)$



old asymp. $y=0 \xrightarrow{-4} y=-4$ new asymp.

b. $f(x) = 5\left(\frac{1}{2}\right)^x + 1$

Basic: $f(x) = \left(\frac{1}{2}\right)^x$

Domain: $\mathbb{R}$
Range: $(1, \infty)$

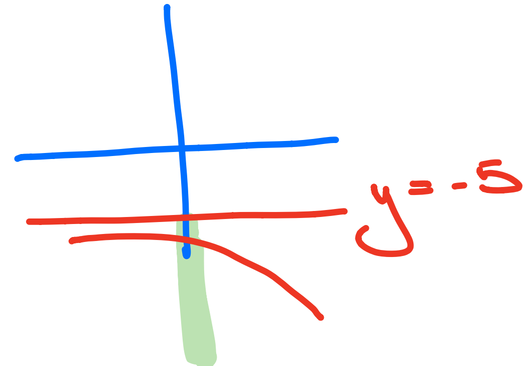
old asymp. $y=0 \xrightarrow{+1} y=1$ new asymp x-refl

c. $f(x) = -3(2^{x+1}) - 5$

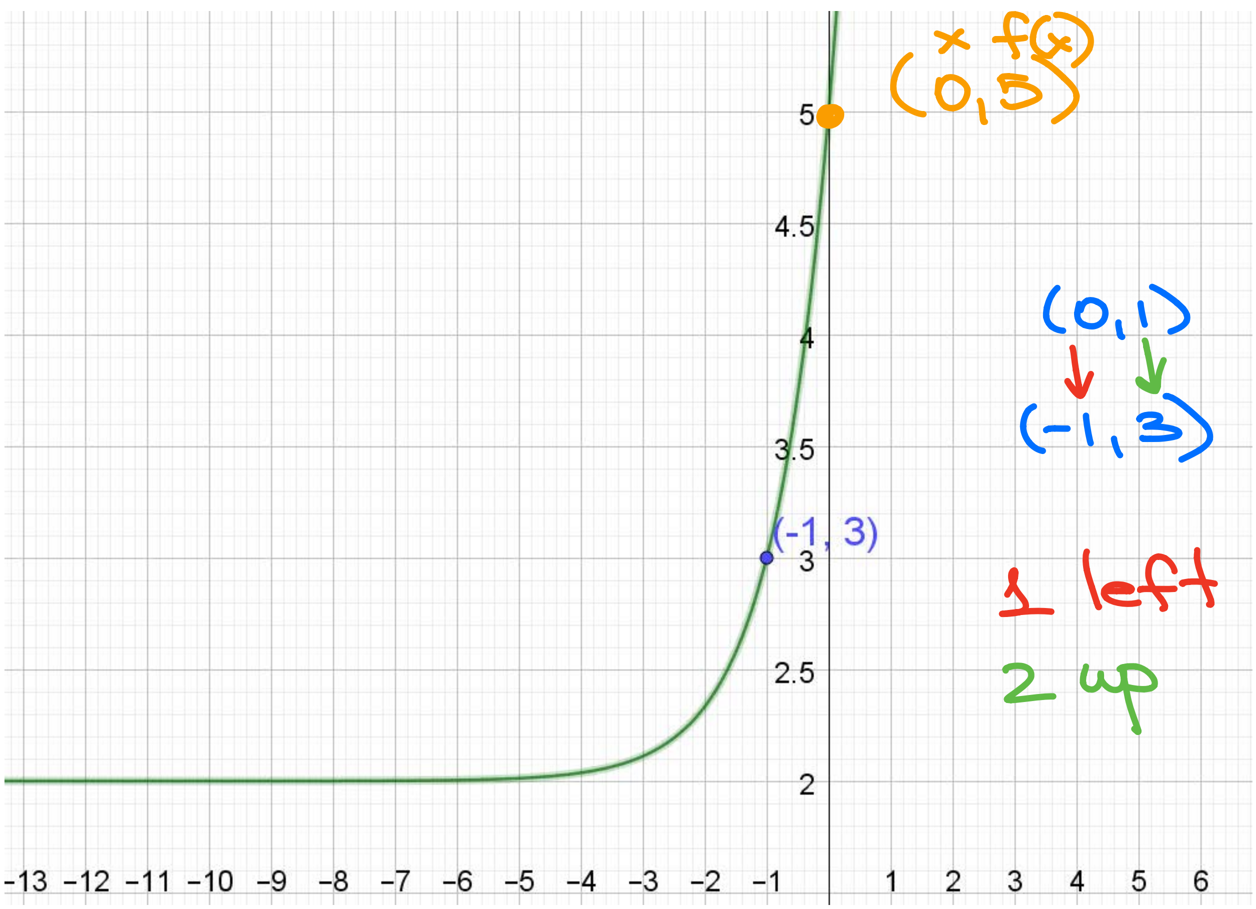
D: $(-\infty, \infty)$

$y = -5$

R: $(-\infty, -5)$



Example: Write an equation of an exponential function with the following graph.



$f(x) = b^{x+1} + 2$

$b^{0+1} + 2 = 5$

$b = 3$

$$\boxed{f(x) = 3^{x+1} + 2}$$

Example: Given $f(x) = 2^{x-1}$ and $g(x) = 5e^{10x}$. Evaluate $(g \circ f)(1)$ and $(g \circ f)(0)$.

$(g \circ f)(1) = g(f(1))$

$f(1) = 2^{1-1} = 2^0 = 1$ $\quad g(1) = 5e^{10(1)} = \boxed{5e^{10}}$

$(g \circ f)(0) = g(f(0))$

$f(0) = 2^{0-1} = 2^{-1} = \frac{1}{2}$ $\quad g(\tfrac{1}{2}) = 5e^{10(1/2)} = \boxed{5e^{5}}$

Logarithmic Functions

Exponential functions are one-to-one; therefore, they are invertible. The inverse function of the exponential function with base b is called the **logarithmic function with base *b***.

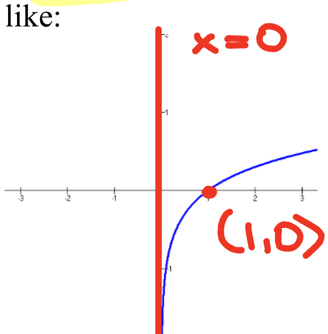
The function $f(x) = \log_b x$ is the **logarithmic function with base *b*** with $x > 0, b > 0$ and $b \neq 1$.

Note: The argument (inside) of a logarithmic function must be positive!

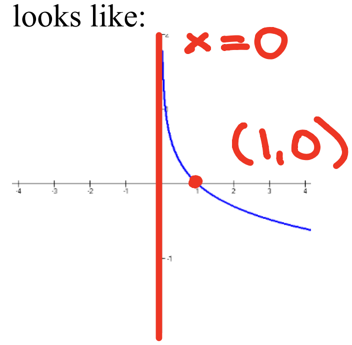
Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Key Point: $(1,0)$

The Graph of a Logarithmic Function

If $b > 1$, the graph of $f(x) = \log_b x$ looks like:



If $0 < b < 1$, the graph of $f(x) = \log_b x$ looks like:



Both graphs have a vertical asymptote of $x = 0$ (the y-axis).

Example: Evaluate $f^{-1}(0)$ if $f(x) = 8^{-1-x} - 32$

$f^{-1}(0) = a$

$f(a) = 0$

$8^{-1-a} - 32 = 0$

$8^{-1-a} = 32$

$(2^3)^{-1-a} = 2^5$

$2^{-3-3a} = 2^5$

$-3-3a = 5$

$-3a = 8$

$a = -8/3$

$\boxed{f^{-1}(0) = -8/3}$

Example: Evaluate $f^{-1}(-1)$ if $f(x) = 5^{1-x} - 2$

$f^{-1}(-1) = a$

$f(a) = -1$

$5^{1-a} - 2 = -1$

$5^{1-a} = 1$

$1 - a = 0$

$a = 1$

$\boxed{f^{-1}(-1) = 1}$

$b \neq 0 \quad b^0 = 1$

Logarithm Rules

$y = \log_b x$ is equivalent to $b^y = x$.

Example: $\log_b 1 = 0$

$b^0 = 1$

For $b > 0, b \neq 1, x > 0, y > 0$:

Inverse Property of Logarithms

1. $\log_b b^x = x$
2. $b^{\log_b x} = x$

The Product Rule

$\log_b(xy) = \log_b x + \log_b y$

The Quotient Rule

$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

The Power Rule

$\log_b x^y = y \log_b x$

$\log x = \log_{10} x$

Example: Evaluate, if possible.

a. $\log_7 49$

$= \log_7 7^2 = 2$

b. $\log 1000$

$= \log_{10} 10^3 = 3$

c. $\log_2(-8)$

DNE

d. $\log 0$

DNE

e. $\ln e^{-3}$ $= -3$

f. $\log_7 \sqrt{7}$

$= \log_7 7^{1/2} = 1/2$

g. $\log_\pi \pi^{-\sqrt{e}}$ $= -\sqrt{e}$

h. $\log_3\left(\frac{1}{9}\right)$ $= \log_3 9^{-1}$

$= -\log_3 9 = -\log_3 3^2 = -2$

Example: Find the domain and vertical asymptotes of the following functions.

a. $\ln(3-2x)$

Domain: $3-2x>0$

$3>2x$

$\frac{3}{2}>x$

$x<3/2$

$(-\infty, 3/2)$

V.A. $3-2x=0$

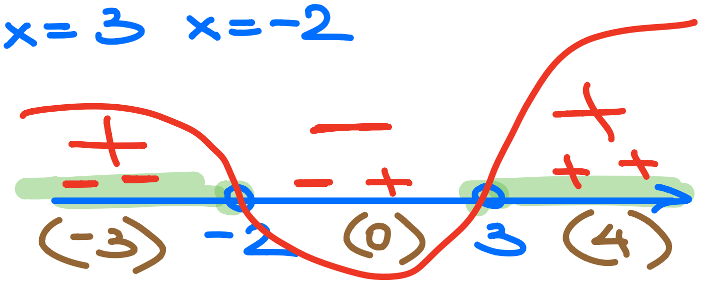
$x=3/2$

b. $\log_3(x^2-x-6)$

$x^2-x-6>0$

$(x-3)(x+2)>0$

$x=3 \quad x=-2$



$D: (-\infty,-2) \cup (3,\infty)$

$x=3 \quad x=-2$

Example: Write the following logarithm as a sum of logarithms with no products, powers or quotients.

$$\ln\left[\frac{x^4(x-5)^3}{\sqrt{x+3}}\right]$$

quotient

product

$= \ln x^4 \cdot (x-5)^3 - \ln (x+3)^{1/2}$

$= \ln x^4 + \ln (x-5)^3 - \ln (x+3)^{1/2}$

$= 4\ln x + 3\ln(x-5) - \frac{1}{2}\ln(x+3)$

Example: Rewrite the following expression as a single logarithm.

$$3\log_7(x-2) - \frac{1}{5}\log_7(x^2-3) - 4\log_7(x+5) + 1$$

$= \log_7(x-2)^3 - \log_7(x^2-3)^{1/5} - \log_7(x+5)^4 + \log_7 7$

$= \left[\log_7(x-2)^3 + \log_7 7\right] - \left[\log_7(x^2-3)^{1/5} + \log_7(x+5)^4\right]$

$= \log_7 7(x-2)^3 - \log_7 \sqrt[5]{x^2-3}\,(x+5)^4$

$= \log_7 \dfrac{7(x-2)^3}{\sqrt[5]{x^2-3}\,(x+5)^4}$

Example: Find $(f \circ g)(x)$ when $f(x) = e^{3x}$ and $g(x) = 5\ln x$

$$f(g(x)) = e^{3(5\ln x)} = e^{\boxed{15}\ln x}$$

$$= e^{\ln x^{15}} = \boxed{x^{15}}$$

Example: Given $f(x) = 5e^{4x-1}$ and $g(x) = \log_2(4x+1)$, find $f^{-1}(5) + g^{-1}(2)$.

$$= 1/4 + 3/4 = \boxed{1}$$

$f^{-1}(5) = a$

$f(a) = 5$

$5e^{4a-1} = 5$

$e^{4a-1} = 1$

$4a - 1 = 0$

$4a = 1$

$a = 1/4$

$f^{-1}(5) = 1/4$

$g^{-1}(2) = a$

$g(a) = 2$

$\log_2(4a+1) = 2$

$2^2 = 4a + 1$

$4 = 4a + 1$

$4a = 3$

$a = 3/4$

$g^{-1}(2) = 3/4$

$\ln x = \log_e x$

Example: Solve for x.

$$32^{\frac{x}{2}} = 20$$

$\ln 32^{\frac{x}{2}} = \ln 20$

$\frac{x}{2}\ln 32 = \ln 20$

$x \ln 32 = 2 \ln 20$

$x = \dfrac{2\ln 20}{\ln 32} = \dfrac{2\ln(2^2 \cdot 5)}{\ln(2^5)}$

$= \dfrac{2\ln 2^2 + 2\ln 5}{5\ln 2} = \boxed{\dfrac{4\ln 2 + 2\ln 5}{5\ln 2}}$

$\log 32^{\frac{x}{2}} = \log 20$

$\dfrac{4\log 2 + 2\log 5}{5\log 2}$

Example: Solve for x.

$$2^{x-1} = \frac{1}{16}$$

$2^{x-1} = \dfrac{1}{2^4}$

$2^{x-1} = 2^{-4}$

$x - 1 = -4$

$\boxed{x = -3}$

Example: Solve for x.

$$\log(4x-1)+3=5$$
$$-3 \quad -3$$

$\log_{10}(4x-1)=2$

$10^2 = 4x-1$

$4x-1 = 100$

$4x = 101$

$$\boxed{x = \frac{101}{4} = 25.25}$$

Check: $4x-1>0$

$4\left(\frac{101}{4}\right)-1>0$

Example: Solve for x.

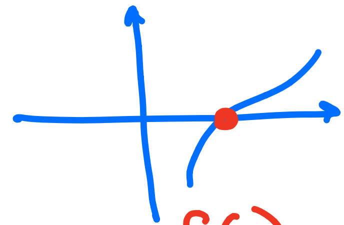
$$\frac{6e^x}{6} = \frac{13}{6}$$

$\ln e^x = \ln \frac{13}{6}$

$x = \ln \frac{13}{6} = \boxed{\ln 13 - \ln 6}$

Example: Find any x- intercepts of the following function.

$$f(x) = \log_6(x+5) + \log_6 x - \log_5 25$$



$f(x)=0$

$$\log_6(x+5) + \log_6 x - \log_5 5^2 = 0$$

$$\log_6 x(x+5) - 2 = 0$$

$$\Downarrow$$

$$\log_6 x(x+5) = 2$$

$$6^2 = x(x+5)$$

$$36 = x^2 + 5x$$

$$x^2 + 5x - 36 = 0$$

$$(x+9)(x-4) = 0$$

$$\cancel{x=-9} \qquad \boxed{x=4}$$

Check:

$x>0$

$x+5>0$