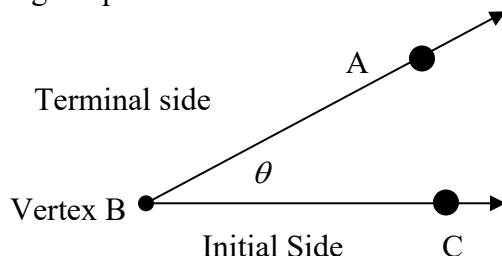


**1330 - Section 4.2**  
**Radians, Arc Length, and Area of a Sector**

Two rays that have a common endpoint (**vertex**) form an **angle**. One ray is the **initial side** and the other is the **terminal side**. We typically will draw angles in the coordinate plane with the initial side along the positive  $x$ -axis.



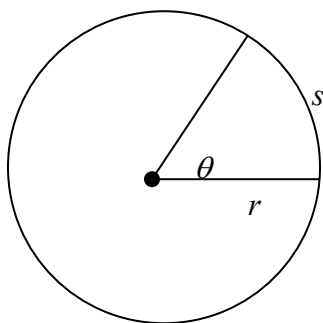
$\angle B$ ,  $\angle ABC$ ,  $\angle CBA$ , and  $\theta$  are all notations for this angle. When using the notation  $\angle ABC$  and  $\angle CBA$  the vertex is always the middle letter.

We measure angles in two different ways, both of which rely on the idea of a complete revolution in a circle. The first is degree measure. In this system of angle measure one complete revolution is  $360^\circ$ . So one degree is  $\frac{1}{360}$  th of a circle.

The second method is called **radian measure**. One complete revolution is  $2\pi$ . The problems in this section are worked in radians. Radians is a unit free measurement.

**The Radian Measure of an Angle**

Place the vertex of the angle at the center of a circle (central angle) of radius  $r$ . Let  $s$  denote the length of the arc intercepted by the angle. The **radian measure**  $\theta$  of the angle is the ratio of the arc length  $s$  to the radius  $r$ . In symbols,  $\theta = \frac{s}{r}$ . In this definition, it is assumed that  $s$  and  $r$  have the same linear units.



If the central angle  $\theta$  and radius  $r$  are given we can use the same formula to calculate the arc length  $s$  by applying the formula:  $s = r\theta$ .

**Example:** A central angle,  $\theta = \frac{\pi}{2}$ , in a circle intercepts an arc of length  $\frac{12\pi}{5}$  m. What is the radius of the circle?

### Relationship between Degrees and Radians

How can we obtain a relationship between degrees and radians? We compare the number of degrees and the number of radians in one complete rotation in a circle. We know that  $360^\circ$  is all the way around a circle. The length of the intercepted arc is equal to the circumference of the circle. Therefore, the radian measure of this central angle is the circumference of the circle divided by the circle's radius,  $r$ . The circumference of a circle of a radius  $r$  is  $2\pi r$ .

We use the formula for radian measure to find the radian measure of the  $360^\circ$  angle.

$$\theta = \frac{s}{r} = \frac{\text{the circumference of a circle}}{r} = \frac{2\pi r}{r} = 2\pi$$

So,  $360^\circ = 2\pi$  radians.

Dividing both sides by 2, we get  $180^\circ = \pi$  radians. Dividing this last equation by  $180^\circ$  or  $\pi$  gives the conversion rules that follow.

### Conversion between Degrees and Radians

1. To convert degrees to radians, multiply degrees by  $\frac{\pi}{180^\circ}$ .
2. To convert radians to degrees, multiply radians by  $\frac{180^\circ}{\pi}$ .

Note: The unit you are converting **to** appear in the **numerator** of the conversion factor.

**Example:** Convert each angle in degrees to radians.

a.  $150^\circ$

b.  $-135^\circ$

**Example:** Convert each angle in radians to degrees.

a.  $\frac{\pi}{3}$

b.  $-\frac{2\pi}{9}$

**Common Angles (Memorize these!)**

$$360^\circ = 2\pi$$

$$90^\circ = \frac{\pi}{2}$$

$$45^\circ = \frac{\pi}{4}$$

$$180^\circ = \pi$$

$$60^\circ = \frac{\pi}{3}$$

$$30^\circ = \frac{\pi}{6}$$

**Sector Area Formula**

In a circle of radius  $r$ , the area  $A$  of a sector with central angle of radian measure  $\theta$  is given by

$$A = \frac{1}{2}r^2\theta$$

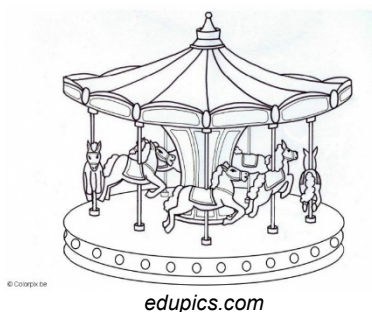
Note:  $\theta$  must be in radian measure!

**Example:** Given the area of sector of a circle is  $\frac{\pi}{3} \text{ in}^2$  and the central angle is  $\frac{\pi}{6}$ , find the radius.

**Example:** Find the perimeter of a sector with central angle  $60^\circ$  and radius 3m.

### Linear and Angular Velocity (Speed)

Consider a merry-go-round



The ride travels in a circular motion. Some of the horses are right along the edge of the merry-go-round, and some are closer to the center. If you are on one of the horses at the edge, you will travel farther than someone who is on a horse near the center. But the length of time that both people will be on the ride is the same. If you were on the edge, not only did you travel farther, you also traveled faster. However, everyone on the merry-go-round travels through the same number of degrees (or radians).

There are two quantities we can measure from this, **angular velocity** and **linear velocity**.

The **angular velocity** of a point on a rotating object is the number of degrees (or radians or revolutions) per unit of time through which the point turns.

This will be the same for all points on the rotating object. We let the Greek letter  $\omega$  (omega) represent angular velocity. Using the definition above,

$$\omega = \frac{\theta}{t}$$

The **linear velocity** of a point on the rotating object is the distance per unit of time that the point travels along its circular path. This distance will depend on how far the point is from the axis of rotation (for example, the center of the merry-go-round). We denote linear velocity by  $v$ . Using the definition above,

$$v = \frac{s}{t}$$

where  $s$  is the arc length ( $s = r\theta$ ).

We can establish a relationship between the two kinds of speed by substituting  $s = r\theta$  into  $v = \frac{s}{t}$ :

**Example:** If the speed of a revolving gear is 25 rpm (revolutions per minute)

- a. Find the number of degrees per minute through which the gear turns.
  
  
  
  
  
  
  
  
  
  
- b. Find the number of radians per minute through which the gear turns.

**Example:** A Ferris wheel rotates 3 times each minute. The passengers sit in seats that are 25 feet from the center of the wheel.

- a. Find the angular speed in units of radians/minute.
  
  
  
  
  
  
  
  
  
  
  
- b. What is the linear velocity of the passengers in the seats?

**Example:** A particle is moving on the perimeter of a circle with radius  $r = 5$  with angular speed of  $\frac{\pi}{4}$  radians per second. After completing 3 full rotations, the particle traveled for 6 more seconds and stopped. What is the length of the total distance the particle traveled?