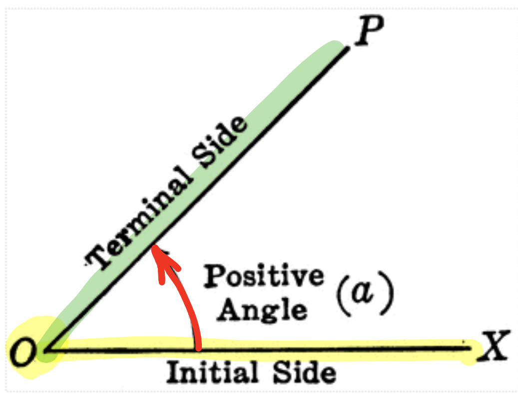
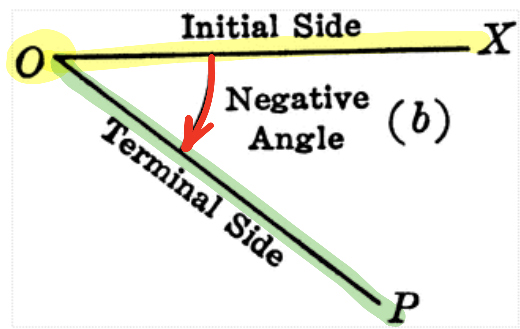
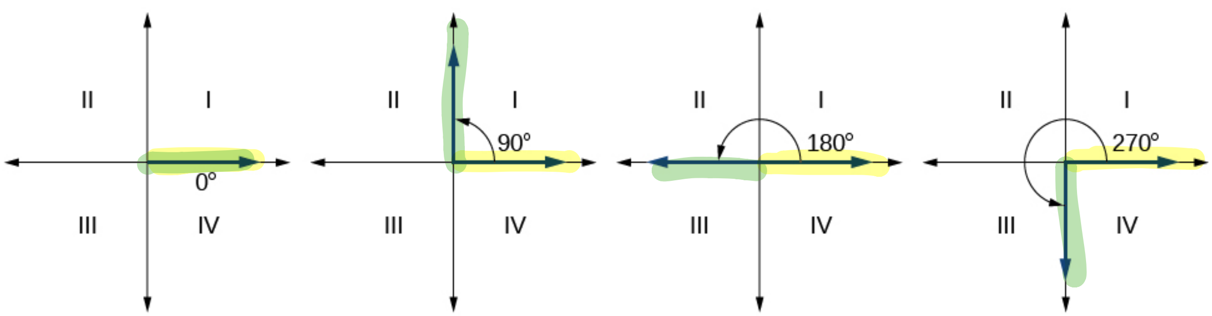
MATH 1330 - Section 4.3 - Trigonometric Functions of Angles

An angle is in **standard position** if the vertex is at the origin of the two-dimensional plane and its initial side lies along the positive *x*-axis. **Positive angles** are generated by counterclockwise rotation. **Negative angles** are generated by clockwise rotation.

An angle in standard position whose terminal side lies on either the x-axis or the y-axis is called a **quadrantal angle**.







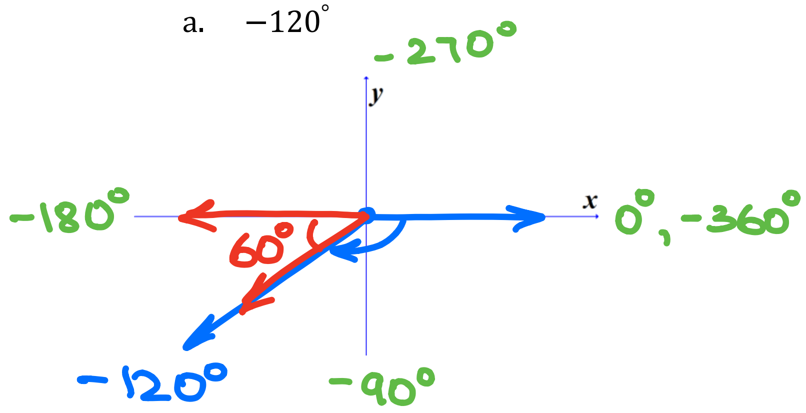
The Reference Angle or Reference Number

Let θ be an angle in standard position. The **reference angle** associated with θ is the acute angle (with positive measure) formed by the x-axis and the terminal side of the angle θ. When radian measure is used, the reference angle is sometimes referred to as the reference number (because a radian angle measure is a real number).

$\frac{7\pi}{4} = 7\left(\frac{\pi}{4}\right)$

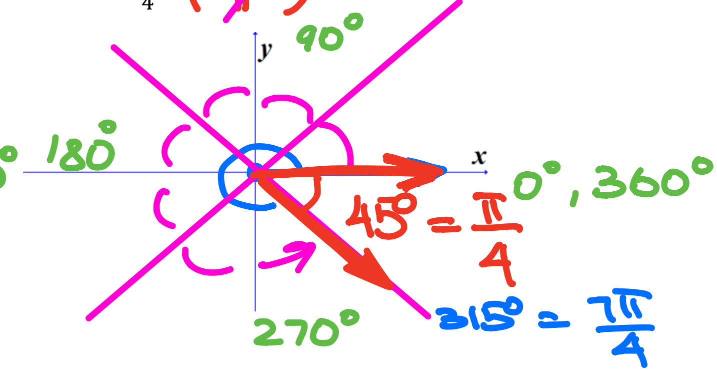
Example: Draw each angle in standard position and specify the reference angle.

a. -120°



Reference angle: 60°

b. $\frac{7\pi}{4}$ $\frac{7\cancel{\pi}}{4}\left(\frac{180^\circ}{\cancel{\pi}}\right) = 315^\circ$



Reference angle: $\frac{\pi}{4}$

Angles that terminate in the exact same position are called **coterminal angles**.
Every angle has infinitely many coterminal angles.
An angle of x° is coterminal with angles $x^\circ + 360^\circ k$, where k is an integer.
An angle of x radians is coterminal with angles $x + 2\pi k$, where k is an integer.

Example: Find a positive and negative angle that is coterminal with $\frac{5\pi}{6}$.

$x + 2\pi k$ $\quad k = \text{integer}$

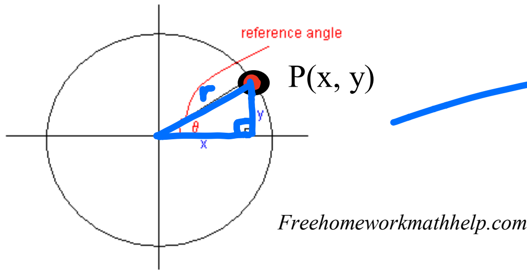
$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

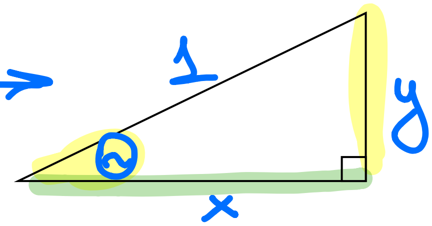
$$\frac{5\pi}{6} + \frac{2\pi \cdot 6}{1 \cdot 6} = \frac{5\pi + 12\pi}{6} = \frac{17\pi}{6}$$

$$\frac{5\pi}{6} - 2\pi = \frac{5\pi - 12\pi}{6} = -\frac{7\pi}{6}$$

We previously defined the six trigonometric functions of an angle as ratios of the length of the sides of a right triangle. Now we will look at them using a **unit** circle centered at the **origin** in the coordinate plane. This circle will have the equation $x^2 + y^2 = 1$.
If we select a point $P(x, y)$ on the circle and draw a ray from the origin though the point, we have created an angle in standard position.

Trigonometric Functions of Angles ***Let us zoom in:***



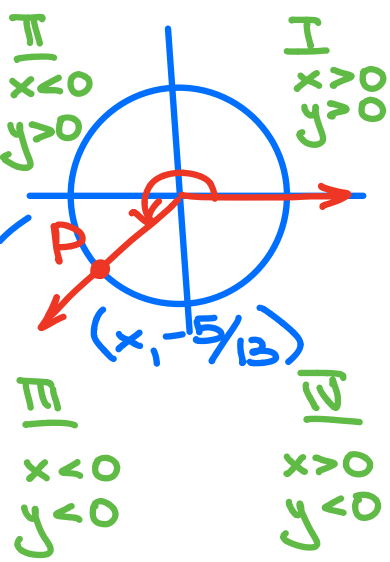


The circle above is the unit circle so $r = 1$. Use the triangle and SOH-CAH-TOA, to obtain the following:

$\cos\theta = x$	$\sin\theta = y$	$\tan\theta = \frac{y}{x},\ x \neq 0$
$\sec\theta = \frac{1}{x},\ x \neq 0$	$\csc\theta = \frac{1}{y},\ y \neq 0$	$\cot\theta = \frac{x}{y},\ y \neq 0$

So for the point $P(x, y) = (\cos\theta, \sin\theta)$.

Example: Let the point $P(x, y)$ denote the point where the terminal side of angle θ (in standard position) meets the unit circle. P is in Quadrant III and $y = -5/13$. Evaluate the six trig functions of θ.



$x^2 + y^2 = 1$

$x^2 + \left(-\frac{5}{13}\right)^2 = 1$

$x^2 + \frac{25}{169} = 1$

$x^2 = 1 - \frac{25}{169}$

$= \frac{169}{169} - \frac{25}{169}$

$= \frac{144}{169}$

$\sqrt{x^2} = -\sqrt{\frac{144}{169}} = -\frac{12}{13}$

$P\left(\overset{x}{-\frac{12}{13}}, \overset{y}{-\frac{5}{13}}\right)$

$\cos\theta = -\frac{12}{13}$

$\sin\theta = -\frac{5}{13}$

$\tan\theta = \frac{y}{x} = -\frac{5}{13} \div \left(-\frac{12}{13}\right)$

$= \frac{5}{\cancel{13}}\left(\frac{\cancel{13}}{12}\right) = \frac{5}{12}$

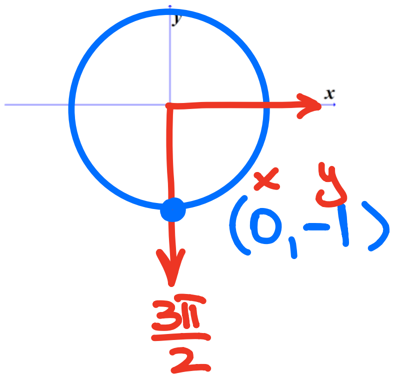
$\sec\theta = -13/12$

$\csc\theta = -13/5$

$\cot\theta = 12/5$

$3\left(\frac{\pi}{2}\right)$

Example: For the quadrantal angle $\frac{3\pi}{2}$, give the coordinates of the point where the terminal side of the angle interests the unit circle. Then find cosine, tangent and cosecant, if possible, of the angle.



$$\cos\left(\frac{3\pi}{2}\right) = 0$$

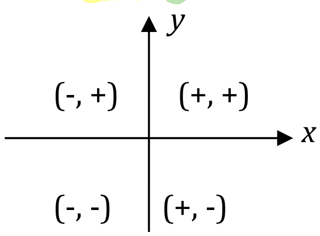
$$\tan\left(\frac{3\pi}{2}\right) = \frac{y}{x} = \frac{-1}{0} \leftarrow \text{BAD!}$$

undefined

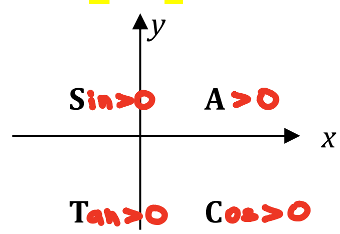
$$\csc\left(\frac{3\pi}{2}\right) = \frac{1}{y} = \frac{1}{-1} = -1$$

Recall: $(x, y) = (\cos\theta, \sin\theta)$

All **S**tudents **T**ake **C**alculus



OR



This should help you to know which trigonometric functions are positive in which quadrant.

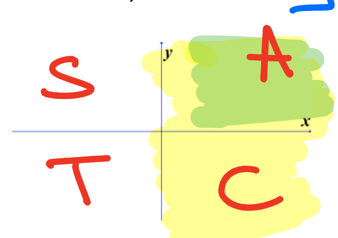
Example: Name the quadrant in which the given conditions are satisfied.

$\cos\theta > 0$
$\tan\theta > 0$

a. $\tan\theta > 0, \sin\theta < 0$

Quadrant III

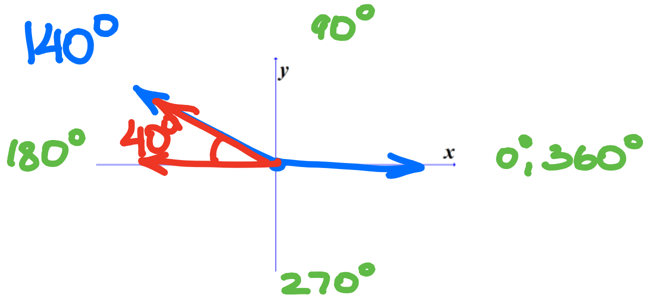
b. $\sec\theta > 0, \cot\theta > 0$



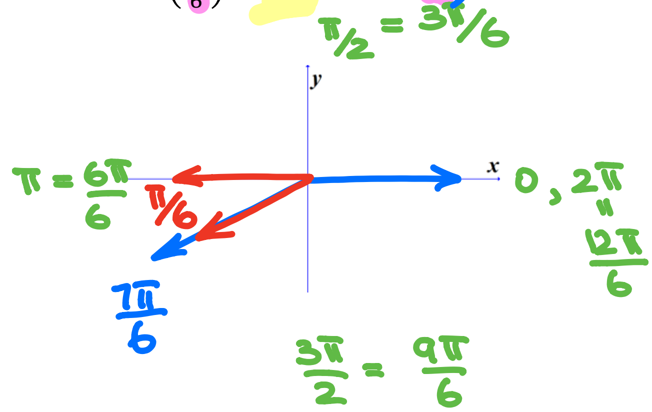
Quadrant I

Example: Rewrite each expression in terms of its reference angle, deciding on the appropriate sign.

a. $\sin(140^\circ) = +\sin(40^\circ)$

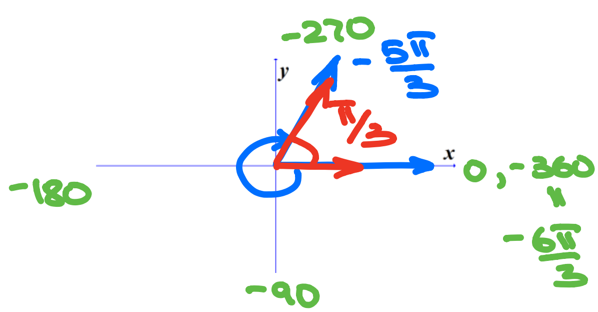


b. $\cos\left(\frac{7\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right)$



5(60°) = −300°

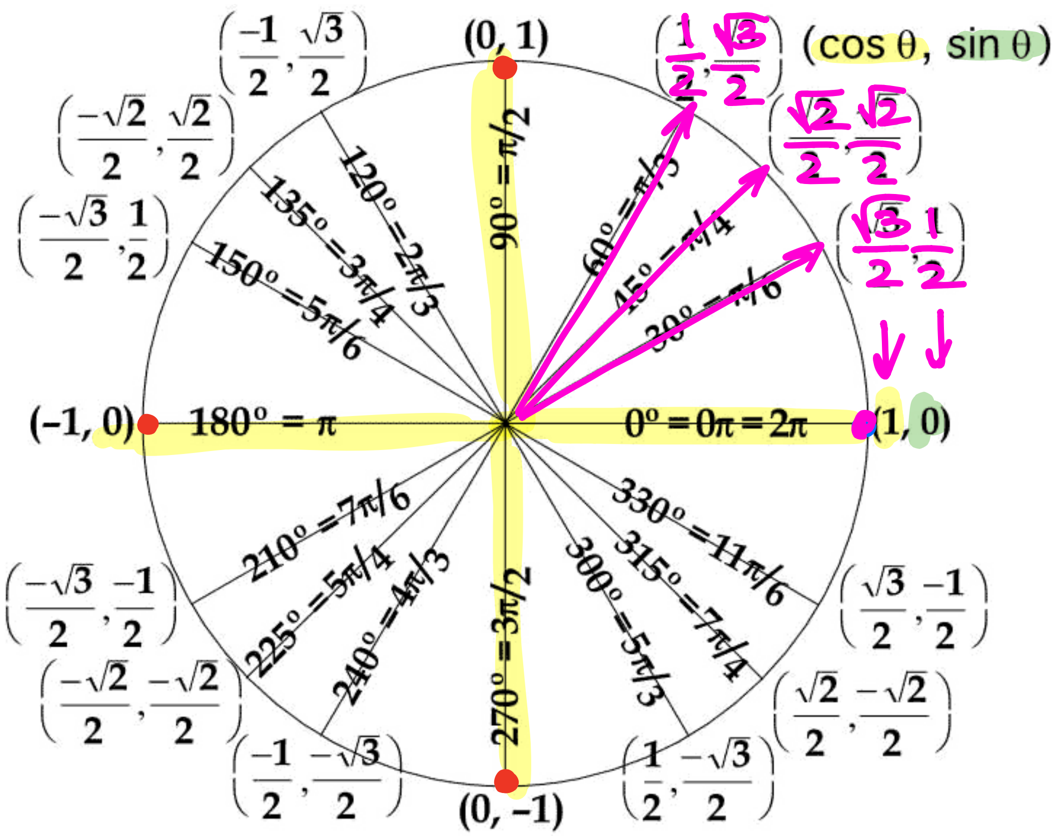
c. $\sec\left(-\frac{5\pi}{3}\right)$



$$\sec\left(-\frac{5\pi}{3}\right) = +\sec\left(\frac{\pi}{3}\right)$$

$$\cos\left(\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right)$$

The Unit Circle



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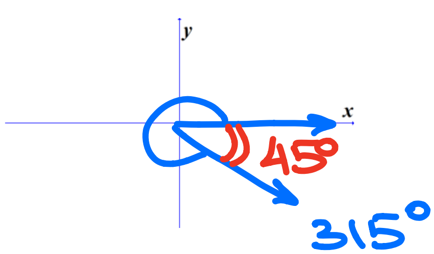
Evaluating Trigonometric Functions Using Reference Angles

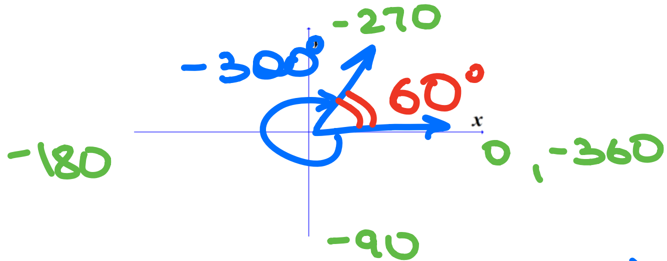
1. Determine the reference angle associated with the given angle.
2. Evaluate the given trigonometric function of the reference angle.
3. Affix the appropriate sign determined by the quadrant of the terminal side of the angle in standard position.

Example: Evaluate the following.

a. $\sin(315^\circ) = -\sin(45^\circ) = -\frac{\sqrt{2}}{2}$

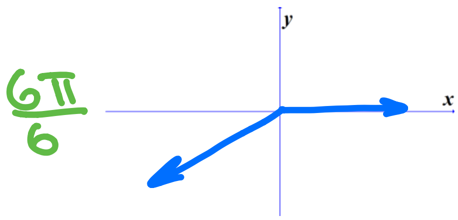


b. $\sec(-300^\circ)$ $= +\sec(60^\circ) = \dfrac{1}{\cos(60^\circ)}$

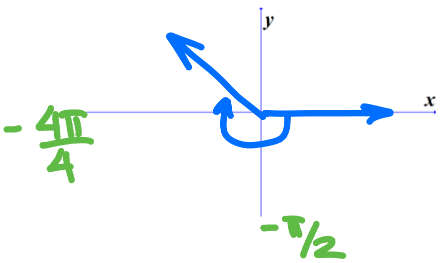


$= \dfrac{1}{\frac{1}{2}} = 2$

c. $\cos\left(\frac{7\pi}{6}\right)$ $= -\cos\left(\frac{\pi}{6}\right) = \boxed{-\frac{\sqrt{3}}{2}}$



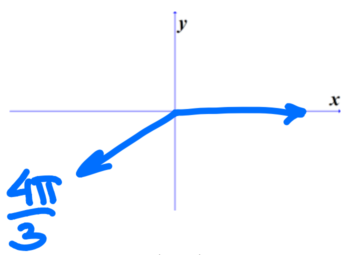
d. $\tan\left(-\frac{5\pi}{4}\right)$ $= -\tan\left(\frac{\pi}{4}\right) = -\dfrac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)}$



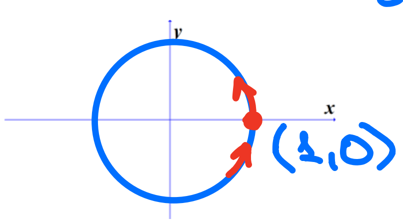
$= -\dfrac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \boxed{-1}$

e. $\sin\left(\frac{4\pi}{3}\right)$ $= -\sin\left(\frac{\pi}{3}\right)$

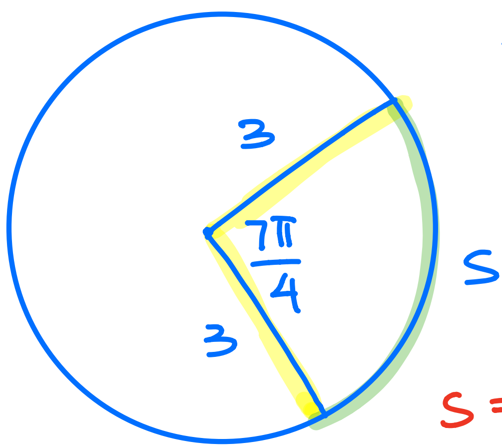
$= \boxed{-\frac{\sqrt{3}}{2}}$



f. $\csc(2\pi)$ $= \dfrac{1}{\sin(2\pi)} = \dfrac{1}{0} \leftarrow$ BAD!



$\boxed{\text{undefined}}$



$$P = 3 + 3 + s$$

$$= 6 + \frac{21\pi}{4}$$

$$s = r\theta = 3\left(\frac{7\pi}{4}\right) = \frac{21\pi}{4}$$