2312 - Section 4.4 - Trigonometric Expressions and Identities

In this section we are going to practice the algebra involved in working with the trigonometric functions.

$f(x)$

Notational Conventions

1. An expression such as $\sin\theta$ really means $\sin(\theta)$.
   An exception to this, however, occurs in expressions such as $\sin(A+B)$, where the parentheses are necessary.
   Example: $\sin\left(\frac{\pi}{4}+\frac{\pi}{2}\right) \neq \sin\frac{\pi}{4}+\frac{\pi}{2}$

2. Parentheses are often omitted in multiplication.
   For example: $(\sin\theta)(\cos\theta)$ is usually written $\sin\theta\cos\theta$.

3. The quantity $(\sin\theta)^n = \sin^n\theta = \underbrace{\sin\theta\sin\theta\ldots\sin\theta}_{n\ times}$.
   Example: $(\sin\theta)^2 = \sin^2\theta = \sin\theta\sin\theta$.
   $\sin^2\theta \neq \sin\theta^2$

In simplifying expressions, it may be useful to use the following identities.

Basic Trigonometric Identities

Reciprocal Identities

1. $\sec\theta = \frac{1}{\cos\theta}, \cos\theta \neq 0$
   $\csc\theta = \frac{1}{\sin\theta}, \sin\theta \neq 0$
   $\cot\theta = \frac{1}{\tan\theta}, \tan\theta \neq 0$

2. $\frac{\sin\theta}{\cos\theta} = \tan\theta\,;\ \frac{\cos\theta}{\sin\theta} = \cot\theta$

Pythagorean Identities

3. $\sin^2\theta + \cos^2\theta = 1$
   $\tan^2\theta + 1 = \sec^2\theta$
   $\cot^2\theta + 1 = \csc^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$1 = \sec^2\theta - \tan^2\theta$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

Example: Simplify.

$$\overset{a}{(\sin\alpha} - \overset{b}{\cos\alpha)}^2 + 2\sin\alpha\cos\alpha$$

Recall: $(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab + b^2$

$$= \sin^2\alpha \cancel{- 2\sin\alpha\cos\alpha} + \cos^2\alpha \cancel{+ 2\sin\alpha\cos\alpha}$$

$$= \sin^2\alpha + \cos^2\alpha = 1$$

FOIL

Example: Simplify.

$$(1-\cos\alpha)(\csc\alpha + \cot\alpha)$$

$$= \csc\alpha + \cot\alpha - \cos\alpha\csc\alpha - \cos\alpha\cot\alpha$$

$$= \frac{1}{\sin\alpha} + \cancel{\frac{\cos\alpha}{\sin\alpha}} - \cancel{\cos\alpha \cdot \frac{1}{\sin\alpha}} - \cos\alpha \cdot \frac{\cos\alpha}{\sin\alpha}$$

$$= \frac{1}{\sin\alpha} - \frac{\cos^2\alpha}{\sin\alpha} = \frac{1-\cos^2\alpha}{\sin\alpha} = \frac{\sin^2\alpha}{\sin\alpha} = \sin\alpha$$

Example: Simplify.

$$\sin^4\theta - 2\sin^2\theta + 1$$

$$a^2 - 2a + 1 = (a-1)^2$$

$$= (\sin^2\theta - 1)^2$$

$$= (-\cos^2\theta)^2$$

$$= \boxed{\cos^4\theta}$$

$\sin^2\theta = a$

$(\sin^2\theta)^2 = \sin^4\theta$
$= a^2$

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + \cos^2\theta - 1 = 0$

$\sin^2\theta - 1 = -\cos^2\theta$

Example: Simplify.

$$\sin\alpha\,(\cot\alpha + \tan\alpha)$$

$= \sin\alpha \cdot \cot\alpha + \sin\alpha \cdot \tan\alpha$

$= \cancel{\sin\alpha} \cdot \dfrac{\cos\alpha}{\cancel{\sin\alpha}} + \sin\alpha \cdot \dfrac{\sin\alpha}{\cos\alpha}$

$= \dfrac{\cos\alpha \cdot \cos\alpha}{1 \cdot \cos\alpha} + \dfrac{\sin^2\alpha}{\cos\alpha} = \dfrac{\cos^2\alpha + \sin^2\alpha}{\cos\alpha} = \dfrac{1}{\cos\alpha}$

$= \boxed{\sec\alpha}$

Example: Simplify.

$$\cos^2\alpha + \sin^2\alpha + \cot^2\alpha$$

$= 1 + \cot^2\alpha = \boxed{\csc^2\alpha}$

Example: Simplify.

$$3\cos^2\alpha + \sec^2\alpha + 3\sin^2\alpha - \tan^2\alpha$$

$= 3(\cos^2\alpha + \sin^2\alpha) + (\sec^2\alpha - \tan^2\alpha)$

$= 3(1) + 1 = \boxed{4}$

$\tan^2\alpha + 1 = \sec^2\alpha$

$1 = \sec^2\alpha - \tan^2\alpha$

$\tan^2\alpha = \sec^2\alpha - 1$

$(a-b)(a+b) = a^2 - b^2$

$LCD = (\sec\alpha+1)(\sec\alpha-1)$

$= \sec^2\alpha - 1 = \tan^2\alpha$

Example: Simplify.

$$\frac{\tan\alpha}{\sec\alpha+1}+\frac{\tan\alpha}{\sec\alpha-1}$$

$$\frac{\tan\alpha}{(\sec\alpha+1)}\cdot\frac{(\sec\alpha-1)}{(\sec\alpha-1)}+\frac{\tan\alpha}{\sec\alpha-1}\cdot\frac{(\sec\alpha+1)}{(\sec\alpha+1)}$$

$$=\frac{\tan\alpha(\sec\alpha-1)+\tan\alpha(\sec\alpha+1)}{(\sec\alpha+1)(\sec\alpha-1)}$$

$$=\frac{\tan\alpha[\sec\alpha-\cancel{1}+\sec\alpha+\cancel{1}]}{\sec^2\alpha-1}$$

$a \div \frac{b}{c}$

$= a\cdot\frac{c}{b}$

$$=\frac{\cancel{\tan\alpha}\cdot 2\sec\alpha}{\tan^{\cancel{2}}\alpha}=\frac{2\sec\alpha}{\tan\alpha}$$

$$=2\cdot\frac{1}{\cos\alpha}\div\frac{\sin\alpha}{\cos\alpha}=\frac{2}{\cancel{\cos\alpha}}\cdot\frac{\cancel{\cos\alpha}}{\sin\alpha}$$

$$=\frac{2}{\sin\alpha}=\boxed{2\csc\alpha}$$

Example: Simplify.

$$1-\frac{\cos^2\alpha}{1-\sin\alpha}$$

$$=\frac{1-\sin\alpha}{1-\sin\alpha}-\frac{\cos^2\alpha}{1-\sin\alpha}$$

$$=\frac{1-\sin\alpha-\cos^2\alpha}{1-\sin\alpha}=\frac{1-\cos^2\alpha-\sin\alpha}{1-\sin\alpha}$$

$$x^2-x=x(x-1)$$

$$=\frac{\sin^2\alpha-\sin\alpha}{1-\sin\alpha}=\frac{\sin\alpha\,(\sin\alpha-1)}{(1-\sin\alpha)}=\boxed{-\sin\alpha}$$

$$=-1$$

$$\frac{a-b}{b-a}=\frac{a-b}{-1(-b+a)}=\frac{(a-b)}{-1(a-b)}=-1$$