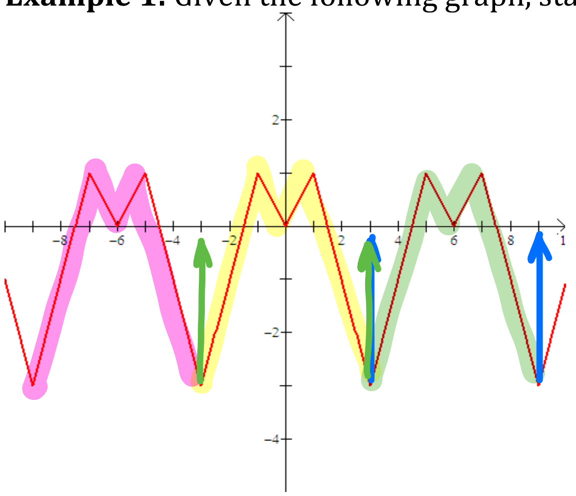
2312 - Section 5.2 – Graphs of Sine and Cosine Functions

A Periodic Function and Its Period

A nonconstant function f is said to be **periodic** if there is a number $p > 0$ such that $f(x+p) = f(x)$ for all x in the domain of f. The smallest such number p is called the period of f.

The graphs of periodic functions display patterns that repeat themselves at regular intervals.

Example 1: Given the following graph, state its period.



$p = 9 - 3 = 6$

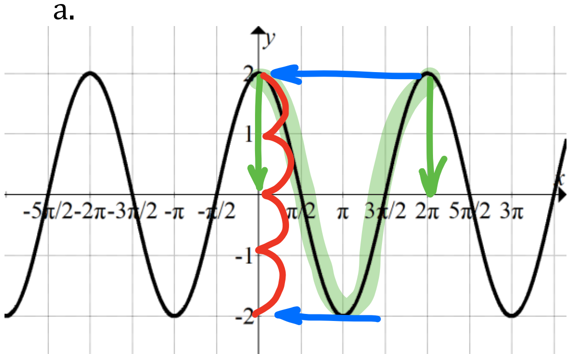
$p = 3 - (-3) = 6$

Amplitude

Let f be a periodic function and let m and M denote, respectively, the minimum and maximum values of the function. Then the **amplitude** of f is the number $\frac{M-m}{2}$.
In other words, the amplitude is half the height.

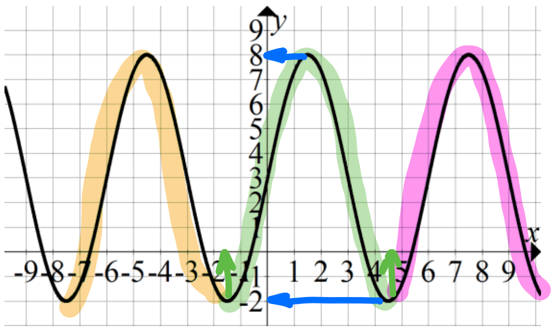
Example 2: Given the following graph, state its amplitude and period.

a.



Amplitude: $\frac{M-m}{2} = \frac{2-(-2)}{2} = 2$ Period: $2\pi - 0 = 2\pi$

b.

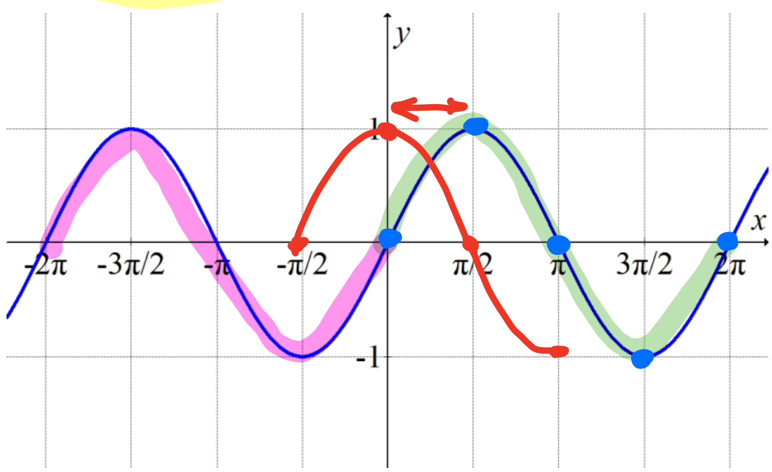


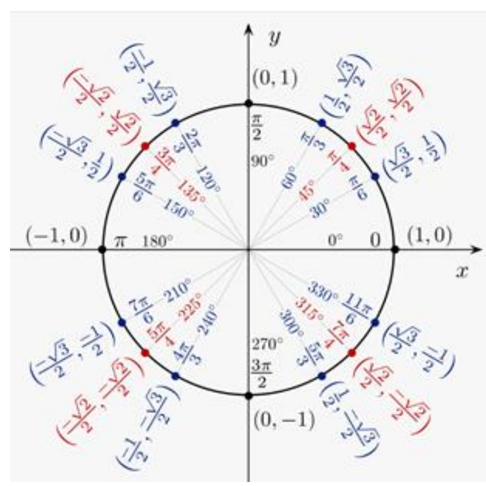
Amplitude: $\frac{M-m}{2} = \frac{8-(-2)}{2} = 5$ Period: $4.5-(-1.5) = 6$

Now let's talk about the graphs of the sine and cosine functions. $\sin(t+\underline{2\pi k}) = \sin(t)$

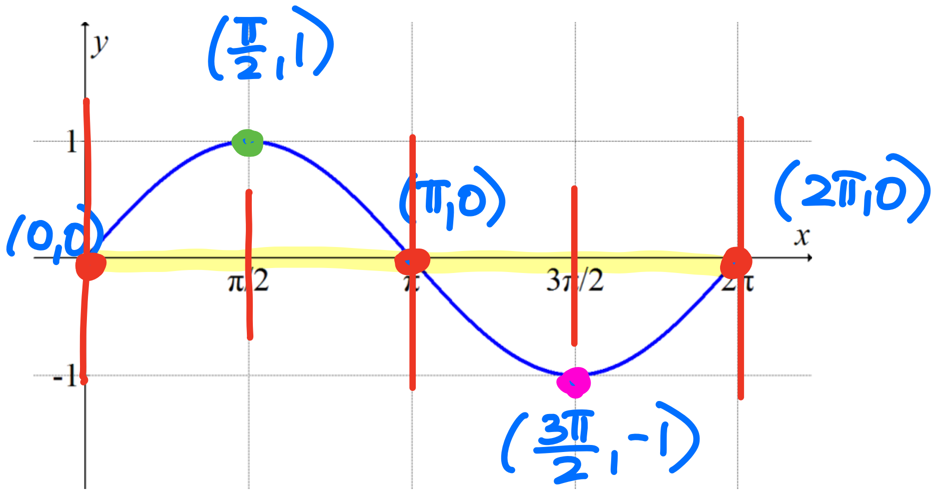
We know from previously studying the periodicity of the trigonometric functions that the sine function repeats itself after 2π radians. Hence, its period is 2π. This also tells us that we can find the whole number line wrapped around the unit circle.

Sine: $f(x) = \sin x$





Since the period of the sine function is 2π, we will graph the function on the interval $[0, 2\pi]$. The rest of the graph is made up of repetitions of this portion.

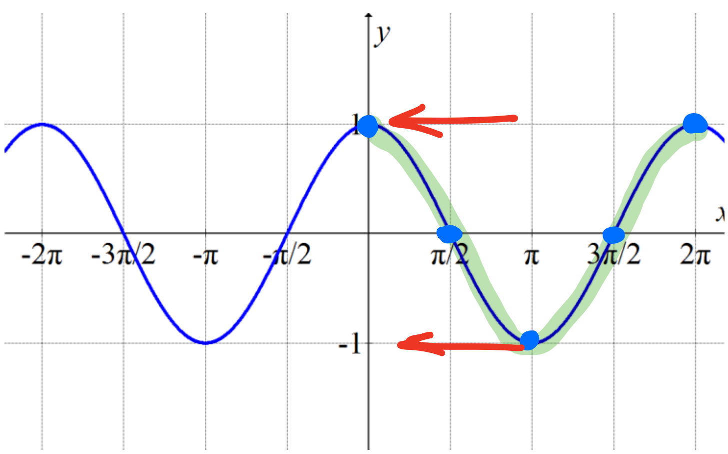


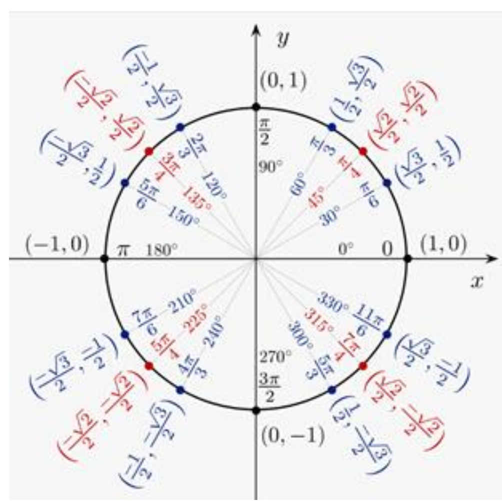
Key points in graphing sine functions are obtained by dividing the period into four equal parts. (Assuming no vertical shifting.)

One complete cycle of the sine curve includes three x-intercepts, one maximum point and one minimum point.

The graph has x-intercepts at the beginning, middle, and end of its full period.

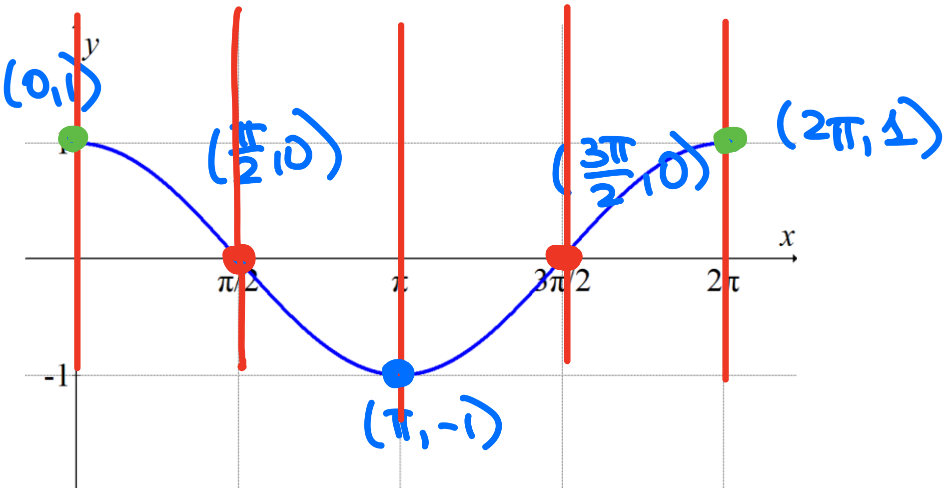
Cosine: $f(x) = \cos x$





Since the period of the cosine function is 2π we will graph the function on the interval $[0, 2\pi]$. The rest of the graph is made up of repetitions of this portion.

$\cos(t + 2\pi k) = \cos(t)$



Key points in graphing cosine functions are obtained by dividing the period into four equal parts. (Assuming no vertical shifting and no x-axis reflection.)

One complete cycle of the cosine curve includes two x-intercepts, two maximum points and one minimum point.

The graph has x-intercepts at the second and fourth points of its full period.

Notice that the graphs of sine and cosine are exactly the same except for a horizontal shift.
For the following functions: $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$.

- Amplitude $= |A|$ (Note: Amplitude is always positive.) If $A > 1$, then it's a vertical stretch by a factor of A. If $0 < A < 1$, then it's a vertical shrink by a factor of A.
- If $B > 1$, then it's a horizontal shrink by a factor of $1/B$. If $0 < B < 1$, then it's a horizontal stretch by a factor of $1/B$.
- Period $= \frac{2\pi}{B}$
- Translation in horizontal direction (called the *phase shift*) $= \frac{C}{B}$
- If $D > 0$, then it's a vertical shift upward. If $D < 0$ then it's a vertical shift downward.
- If the coefficient of the function is negative, this refers to an x-axis reflection.
- The graphs of $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$ complete one cycle from $x = \frac{C}{B}$ to $x = \frac{C}{B} + \frac{2\pi}{B}$.

$A = \frac{1}{2}$ $B = \frac{\pi}{4}$ $C = -\frac{1}{4}$ $D = 1$

Example 3: State the transformations for: $f(x) = \frac{1}{2}\sin\left(\frac{\pi}{4}x + \frac{1}{4}\right) + 1$

1. $|A| = \frac{1}{2}$ $0 < \frac{1}{2} < 1$ vertical shink by $\frac{1}{2}$
2. $\frac{C}{B} = -\frac{1}{4} \div \frac{\pi}{4} = -\frac{1}{4} \cdot \frac{4}{\pi} = -\frac{1}{\pi}$ shift left by $\frac{1}{\pi}$
3. $\frac{2\pi}{B} = 2\pi \div \frac{\pi}{4} = \cancel{2}\pi \cdot \frac{4}{\cancel{\pi}} = 8$ 4. $D = 1$ up 1

Example 4: Sketch the graph of $f(x) = 3\sin\left(\frac{1}{2}x\right) - 3$. Label, with ordered pairs, any x-intercept(s), y-intercept, maxima, and minima. Then state the domain and range of the function in interval notation.

$A = 3$ $B = \frac{1}{2}$ $C = 0$ $D = -3$

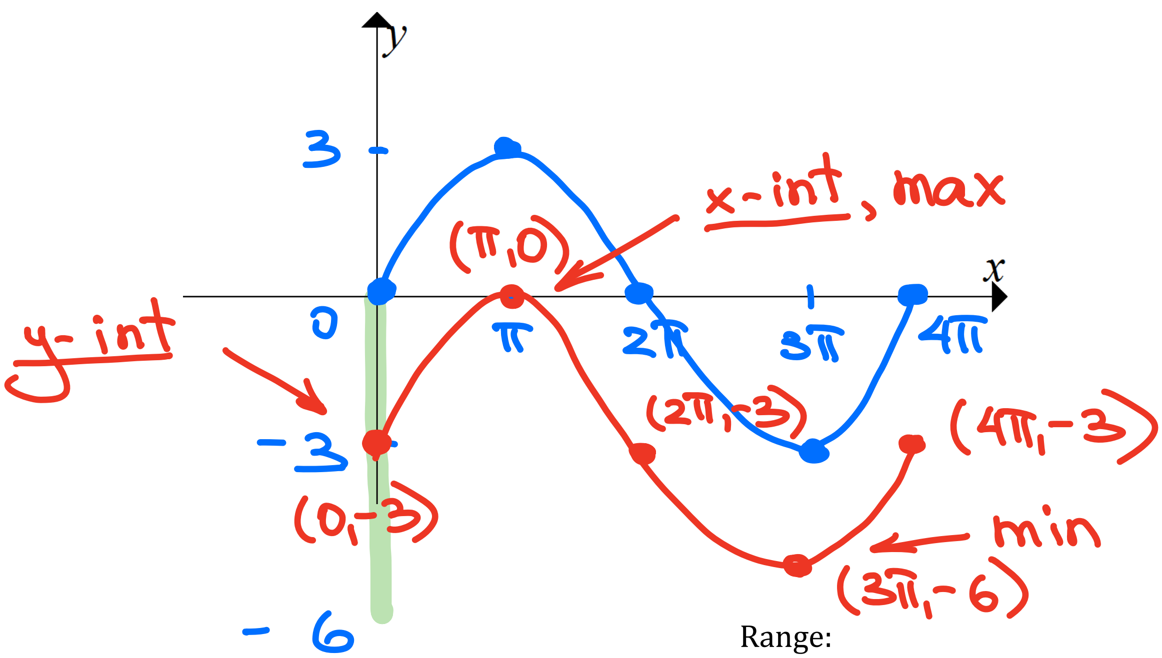
Amplitude: $|A| =$ $|3| = 3$

Period: $\frac{2\pi}{B} =$ $\frac{2\pi}{1/2} = 2\pi \cdot 2 = 4\pi$

Phase Shift: $\frac{C}{B} =$ $\frac{0}{1/2} = 0$

One cycle begins at the phase shift and ends at: $\frac{C}{B} + \frac{2\pi}{B}$ $\longrightarrow [0, 0 + 4\pi]$

Any other transformations? 3 down



Domain:

$(-\infty, \infty)$

Range:

$[-6, 0]$

$A=-1 \quad B=2 \quad C=-\frac{\pi}{2} \quad D=0$

Example 5: Sketch the graph of $f(x) = -\cos\left(2x + \frac{\pi}{2}\right)$. Label, with ordered pairs, any x-intercept(s), y-intercept, maxima, and minima. Then state the domain and range of the function in interval notation.

Amplitude: $|A| = |-1| = 1$

Period: $\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

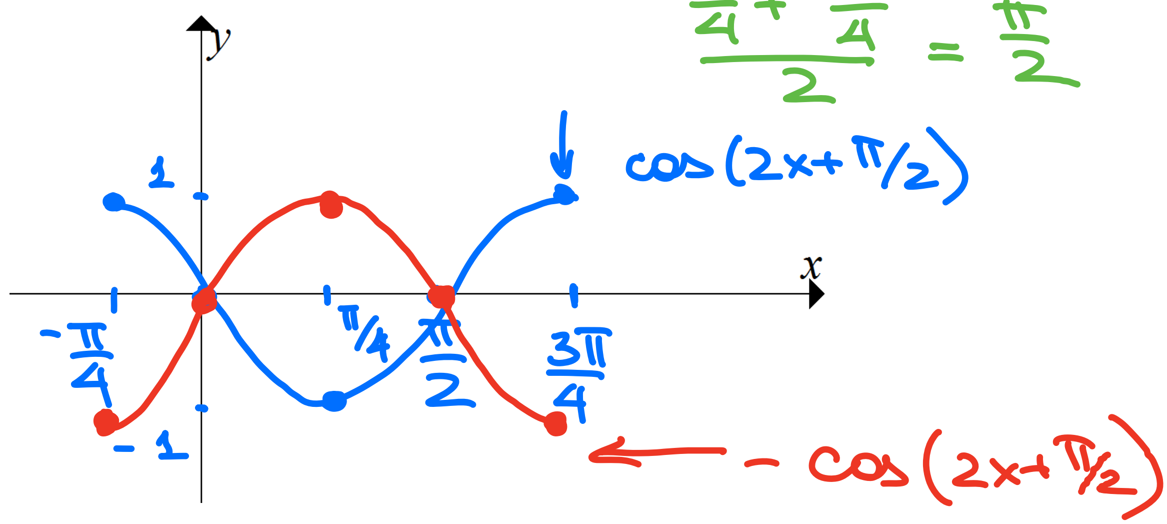
Phase Shift: $\frac{C}{B} = -\frac{\pi}{2} \div 2 = -\frac{\pi}{2} \cdot \frac{1}{2} = -\frac{\pi}{4}$ left

One cycle begins at the phase shift and ends at: $\frac{C}{B} + \frac{2\pi}{B}$ $\quad -\frac{\pi}{4}, \; -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

Any other transformations? x-refl.

$$\frac{-\frac{\pi}{4} + \frac{3\pi}{4}}{2} = \frac{\frac{2\pi}{4}}{2} = \frac{\pi}{4}$$

$$\frac{\frac{\pi}{4} + \frac{3\pi}{4}}{2} = \frac{\pi}{2}$$



Domain: $(-\infty, \infty)$ Range: $[-1, 1]$

Try this one: Sketch the graph of $f(x) = -2\cos\left(\frac{\pi x}{3}\right)$. Label, with ordered pairs, any x-intercept(s), y-intercept, maxima, and minima. Then state the domain and range of the function in interval notation.

$A=-2 \quad B=\frac{\pi}{3} \quad C=0 \quad D=0$

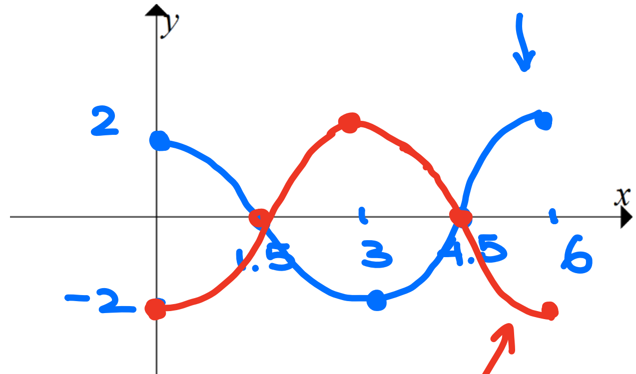
$|A| = |-2| = 2$

$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{\pi/3} = 2\pi \cdot \frac{3}{\pi} = 6$

$\text{Phase shift} = \frac{C}{B} = \frac{0}{\pi/3} = 0$

Key points : 0, 1.5, 3, 4.5, 6

$2\cos\left(\frac{\pi x}{3}\right)$



$-\cos\left(\frac{\pi x}{3}\right)$

Domain: $(-\infty, \infty)$

Range: $[-2, 2]$

Try this one: Sketch the graph of $f(x) = \sin\left(\frac{1}{4}x - \frac{\pi}{8}\right)$. Label, with ordered pairs, any x-intercept(s), y-intercept, maxima, and minima. Then state the domain and range of the function in interval notation.

.

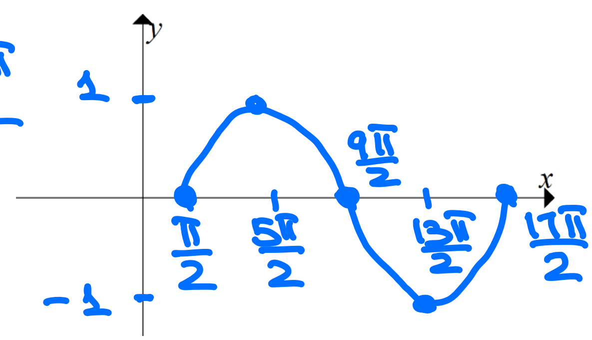
$A=1 \quad B=1/4 \quad C=\pi/8 \quad D=0$

$|A| = |1| = 1$

$\text{Period} = \frac{2\pi}{B} = 2\pi \div \frac{1}{4} = 2\pi \cdot 4 = 8\pi$

$\text{Phase shift} : \frac{C}{B} = \frac{\pi}{8} \div \frac{1}{4} = \frac{\pi}{8} \cdot 4 = \frac{\pi}{2}$

Key points: $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}$

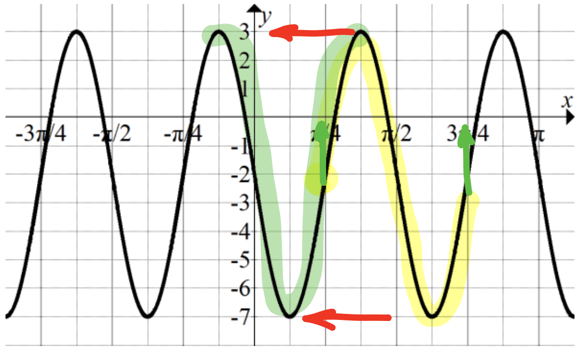


Domain: $(-\infty, \infty)$

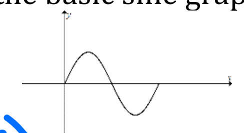
Range: $[-1, 1]$

Example 6: Give an equation of the form $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$ and that could represent the given graph.

ANSWER is NOT UNIQUE!



Sine:
Let's begin by recalling one cycle of the basic sine graph. Then choose one cycle on the graph above.



$5\sin(4x - \pi) - 2$

Amplitude: $A = \frac{M-m}{2} = \frac{3-(-7)}{2} = 5$

Vertical Shift, D: It'll be half-way between the maximum and the minimum values. $D = -2$

Use the period to find B: Recall the period formula $\frac{2\pi}{B} = \left(\frac{3\pi}{4} - \frac{\pi}{4}\right)$ $\quad \frac{2\pi}{B} = \frac{\pi}{2}$ $\quad B\pi = 4\pi$ $\quad B = 4$

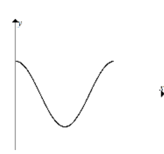
Using the cycle chosen, it begins at 0 so $C = 0$. Recall the one basic cycle of sine, compare it to your chosen cycle. Any other transformations?

$\frac{C}{B} = \frac{\pi}{4}$ $\quad \frac{C}{4} = \frac{\pi}{4}$ $\quad C = \pi$

Sine Function:

$f(x) = 5\sin(4x - \pi) - 2$

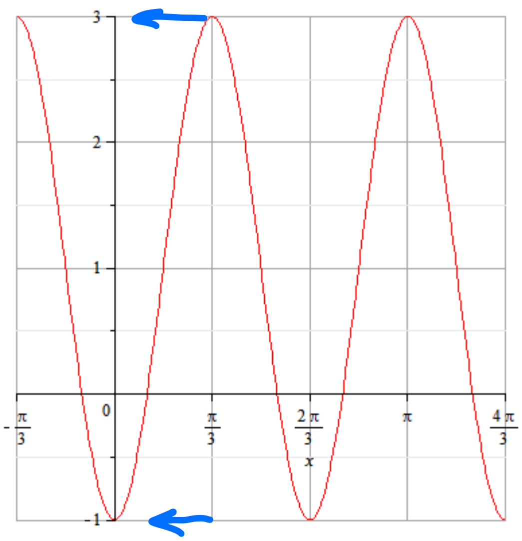
Cosine:
Let's begin by recalling one cycle of the basic cosine graph. Then choose one of these cycles on the graph above.



A, D, B are the same as above. Now compare your chosen cycle with the one basic cycle of cosine. Any other transformations?

Cosine Function: $f(x) = 5\cos\left(4x - \frac{\pi}{2}\right) - 2$

Try this one: Give an equation of the form $y = A\cos(Bx - C) + D$ which could be used to represent the given graph. (Note: C or D may be zero.)



$A = \frac{3-(-1)}{2} = 2$

$D = 1$

a. $f(x) = 2\cos(3x-\pi)-1$

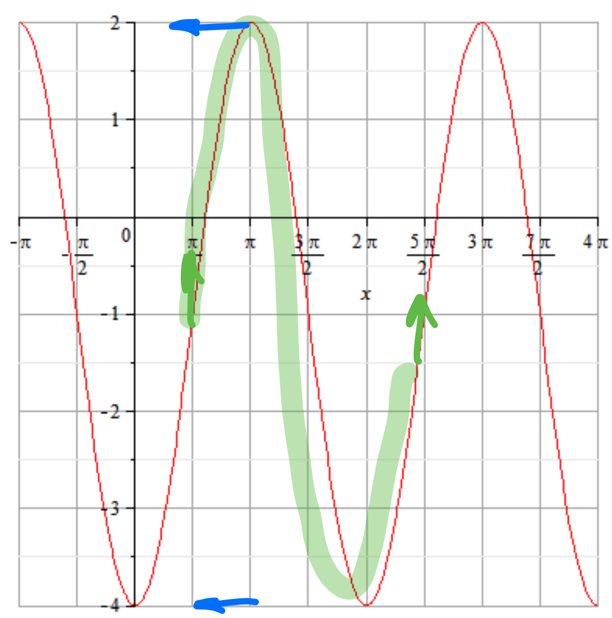
b. $f(x) = 2\cos(3x-\pi)$

c. $\boxed{f(x) = 2\cos(3x-\pi)+1}$

d. $f(x) = 4\cos(3x-\pi)-1$

e. $f(x) = 4\cos(3x-\pi)+1$

Try this one: Give an equation of the form $y = A\sin(Bx - C) + D$ which could be used to represent the given graph. (Note: C or D may be zero.)



$A = \frac{2-(-4)}{2}$

$= 3$

$D = -1$

$\frac{2\pi}{B} = 2\pi$

$B = 1$

$\frac{C}{B} = \frac{\pi}{2}$

$\frac{C}{1} = \frac{\pi}{2}$

$C = \frac{\pi}{2}$

a. $f(x) = 3\sin\left(x - \frac{\pi}{2}\right)$

b. $f(x) = 6\sin\left(x - \frac{\pi}{2}\right)+1$

c. $f(x) = 6\sin\left(x - \frac{\pi}{2}\right)-1$

d. $\boxed{f(x) = 3\sin\left(x - \frac{\pi}{2}\right)-1}$

e. $f(x) = 3\sin\left(x - \frac{\pi}{2}\right)+1$

Models

As we know, trigonometric functions repeat their behavior. Breathing normally, brain waves during deep sleep are just a couple of examples that can be described using a sine function.

Example 7: The current I, in amperes, flowing through an ac (alternating current) circuit at time t is modeled by $I = 210\sin\left(40\pi t - \frac{\pi}{3}\right)$ where $t \geq 0$. Find its:

$B = 40\pi \quad C = \frac{\pi}{3}$

a. Amplitude $A =$ 210

b. Period $\frac{2\pi}{B} = \frac{2\pi}{40\pi} = \frac{1}{20}$

c. Horizontal shift $\frac{C}{B} = \frac{\pi}{3} \div 40\pi = \frac{\cancel{\pi}}{3} \cdot \frac{1}{40\cancel{\pi}} = \frac{1}{120}$

d. Maximum value 210