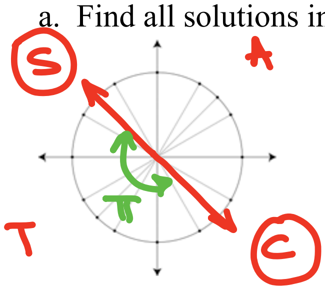
# Section 6.3
# Solving Trigonometric Equations

An equation that contains a trigonometric expression is called a **trigonometric equation**. As we know trigonometric functions repeat their behavior. What if we wanted answers to questions like: "When will the moon look exactly like it did last night at 9:30pm?" and "When will my breathing be exactly as it is right now?". The questions to these models can be answered by solving trigonometric equations.

Example 1: Let $\tan(x) = -1$. Period of $\tan x = \pi$

a. Find all solutions in the interval $[0, 2\pi)$




$$\boxed{x = \frac{3\pi}{4} \quad \text{or} \quad x = \frac{7\pi}{4}}$$

b. Find all solutions.

$$\boxed{x = \frac{3\pi}{4} + k\pi \qquad k = \text{integer}}$$

Example 2: Let $2\sin^2 x - 3\sin x = -1$.

a. Solve the equation in the interval $[0, 2\pi)$.

Think of $2t^2 - 3t = -1$

$2t^2 - 3t + 1 = 0$

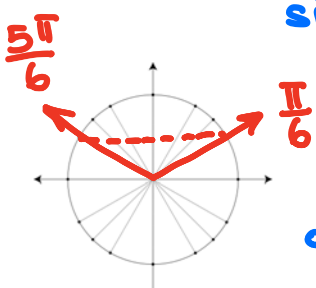
$(2t-1)(t-1) = 0$

$(2\sin x - 1)(\sin x - 1) = 0$

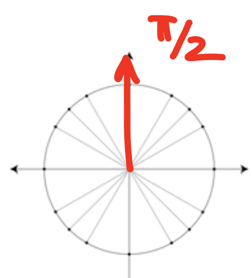
$2\sin x - 1 = 0$ or $\sin x - 1 = 0$

$2\sin x = 1$ $\qquad$ $\sin x = 1$

$\sin x = \frac{1}{2}$




$$\boxed{x = \frac{\pi}{6} \quad \text{or} \quad x = \frac{5\pi}{6}}$$




$$\boxed{x = \frac{\pi}{2}}$$

b. Find all solutions.

$$\boxed{\begin{aligned} x &= \frac{\pi}{6} + 2\pi k \\ x &= \frac{5\pi}{6} + 2\pi k \\ x &= \frac{\pi}{2} + 2\pi k \\ & k = \text{integer} \end{aligned}}$$

$\left[0, \frac{\pi}{2}\right) \Rightarrow$ no solution $\qquad \left[0, 2\pi\right) \Rightarrow x = \frac{\pi}{2}$

$x = \frac{3\pi}{2}$

Example 3: Solve the equation $\sin(2x) = -15\cos x$ in the interval $\left[0, \frac{\pi}{2}\right]$.

*Recall:* $\sin(2x) = 2\sin x\cos x$

$2\sin x\cos x = -15\cos x$

$2\sin x\cos x + 15\cos x = 0$

$\cos x\left[2\sin x + 15\right] = 0$

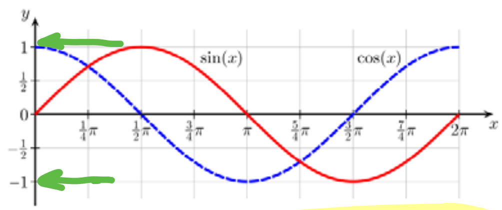
$\cos x = 0$ or $2\sin x + 15 = 0$

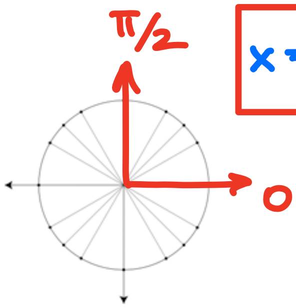
$\boxed{x = \frac{\pi}{2}}$

$\sin x = -\frac{15}{2}$

$\sin x = -7.5$

No solution




*In solving trig equations if you ever get*

- $\sin x > 1$
- $\sin x < -1$
- $\cos x > 1$
- $\cos x < -1$

*then you have no solution.*

Try this one: Solve the equation $10\sin^2 x = 10\cos^2 x$ in the interval $[0, 2\pi)$.

*Recall:* $\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$10\sin^2 x - 10\cos^2 x = 0$

$\sin^2 x - \cos^2 x = 0$

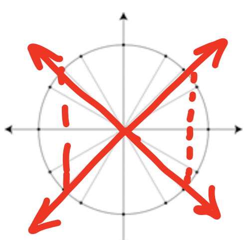
$1 - \cos^2 x - \cos^2 x = 0$

$1 - 2\cos^2 x = 0$

$1 = 2\cos^2 x$

$\sqrt{\cos^2 x} = \pm\sqrt{\frac{1}{2}}$

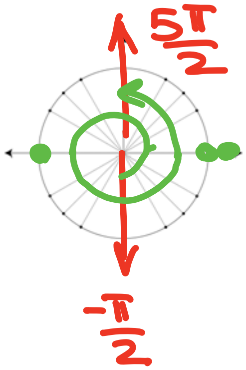
$\cos x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$



$$\boxed{\begin{aligned} x &= \frac{\pi}{4} \\ x &= \frac{3\pi}{4} \\ x &= \frac{5\pi}{4} \\ x &= \frac{7\pi}{4} \end{aligned}}$$

Example 4: Find all solutions of $\sec^2 x = 1$ in the interval $\left[-\frac{\pi}{2}, \frac{5\pi}{2}\right]$.

*Recall:* $\sin^2 x + \cos^2 x = 1$ divided by $\cos^2 x$ produces $\tan^2 x + 1 = \sec^2 x$




$\tan^2 x + 1 = 1$

$\sqrt{\tan^2 x} = \sqrt{0}$

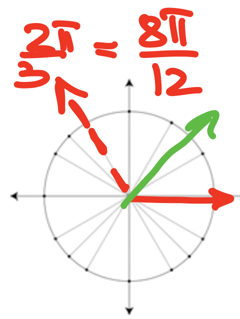
$\tan x = 0$

$\frac{\sin x}{\cos x} = 0$ $\qquad \sin x = 0$

$\boxed{x = 0, \quad x = \pi, \quad x = 2\pi}$

**Equations Involving Multiples of an Angle**

Example 5: Find all solutions of $\tan(3x) = -1$ in the interval $\left[0, \frac{2\pi}{3}\right)$.




$\frac{3}{3}x = \left(\frac{3\pi}{4} + \pi k\right)\frac{1}{3}$

$x = \frac{\cancel{3}\pi}{4} \cdot \frac{1}{\cancel{3}} + \frac{\pi k}{3}$

$x = \frac{\pi}{4} + \frac{\pi k}{3}$

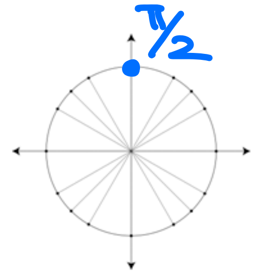
$k = 0 \qquad \boxed{x = \frac{\pi}{4}}$

$k = 1 \qquad x = \frac{\pi}{4} + \frac{\pi}{3} = \frac{3\pi + 4\pi}{12} = \boxed{\frac{7\pi}{12}}$

$k = 2 \qquad x = \frac{\pi}{4} + \frac{2\pi}{3}$ out

$k = -1 \qquad x = \frac{\pi}{4} - \frac{\pi}{3}$ out

Example 6: Find all solutions in $\left[-\frac{1}{3}, \frac{2}{3}\right]$ for $\sin\left(\pi x + \frac{\pi}{3}\right) = 1$




$$\pi x + \frac{\pi}{3} = \frac{\pi}{2} + 2\pi k$$

$$\pi x = \frac{3 \cdot \pi}{3 \cdot 2} - \frac{2 \cdot \pi}{2 \cdot 3} + 2\pi k$$

$$\pi x = \frac{3\pi - 2\pi}{6} + 2\pi k$$

$$\frac{\pi x}{\pi} = \left(\frac{\pi}{6} + 2\pi k\right)\frac{1}{\pi}$$

$$x = \frac{1}{6} + 2k \qquad k = \text{int}$$

$$\left[-\frac{1}{3}, \frac{2}{3}\right] = \left[-\frac{2}{6}, \frac{4}{6}\right]$$

$k = 0$ $\quad \boxed{x = \frac{1}{6}}$

$k = 1$ $\quad x = \frac{1}{6} + 2$ out

$k = -1$ $\quad x = \frac{1}{6} - 2 = \frac{1}{6} - \frac{12}{6} = \frac{-11}{6}$ out

Try these:

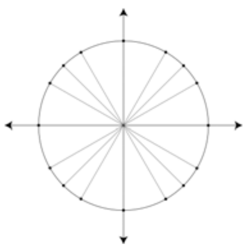
I. Find all solutions of $\frac{\boxed{2\sin\left(\frac{x}{2}\right)}}{2} = \frac{-1}{2}$ in the interval $[-\pi, \pi]$.

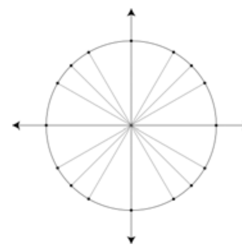
$\sin\left(\frac{x}{2}\right) = -\frac{1}{2}$ Think of $\sin(\ ) = -\frac{1}{2}$

$2\frac{x}{2} = \left(\frac{7\pi}{6} + 2\pi k\right)2$ $\qquad$ $2\frac{x}{2} = \left(\frac{11\pi}{6} + 2\pi k\right)2$

$x = \frac{7\pi}{3} + 4\pi k$ $\qquad$ $x = \frac{11\pi}{3} + 4\pi k$

$[-\pi, \pi] = \left[-\frac{3\pi}{3}, \frac{3\pi}{3}\right]$





$k=0 \quad x = \frac{7\pi}{3}$ out

$k=1 \quad x = \frac{7\pi}{3} + 4\pi$ out

$k=-1 \quad x = \frac{7\pi}{3} - 4\pi$

$= \frac{7\pi}{3} - \frac{12\pi}{3}$

$= -\frac{5\pi}{3}$ out

$k=0 \quad x = \frac{11\pi}{3}$ out

$k=1 \quad x = \frac{11\pi}{3} + 4\pi$ out

$k=-1 \quad x = \frac{11\pi}{3} - 4\pi$

$= \frac{11\pi}{3} - \frac{12\pi}{3}$

$= \boxed{-\frac{\pi}{3}}$

$k = -2$

$x = \frac{11\pi}{3} - 8\pi$

$= \frac{11\pi}{3} - \frac{24\pi}{3} = -\frac{13\pi}{3}$ out

II. Find all solutions in $\left[0, \frac{\pi}{2}\right]$ for $\frac{2\cos\left(2x - \frac{\pi}{6}\right)}{2} = \frac{\sqrt{3}}{2}$

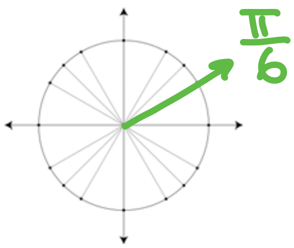
$\cos\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\cos(\quad) = \frac{\sqrt{3}}{2}$

$2x - \frac{\pi}{6} = \frac{\pi}{6} + 2\pi k$

$2x = \frac{\pi}{6} + \frac{\pi}{6} + 2\pi k$

$\frac{2x}{2} = \left(\frac{\pi}{3} + 2\pi k\right)\frac{1}{2}$




$\underline{\left[0, \frac{\pi}{2}\right]} = \left[0, \frac{3\pi}{6}\right]$

$x = \frac{\pi}{6} + \pi k$

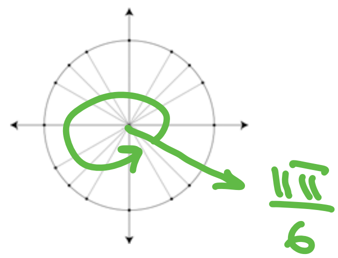
$k = 0 \quad \boxed{x = \frac{\pi}{6}}$

$k = 1 \quad x = \frac{\pi}{6} + \pi$ out

$k = -1 \quad x = \frac{\pi}{6} - \pi$ out

$2x - \frac{\pi}{6} = \frac{11\pi}{6} + 2\pi k$

$2x = \frac{11\pi}{6} + \frac{\pi}{6} + 2\pi k$




$2x = \frac{12\pi}{6} + 2\pi k$

$2x = 2\pi + 2\pi k$

$x = \pi + \pi k$

$k = 0 \quad x = \pi$ out

$k = 1 \quad x = \pi + \pi$ out

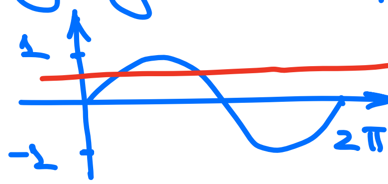
$k = -1 \quad x = \pi - \pi = \boxed{0}$

Solutions: $x = \frac{\pi}{6}$, $x = 0$

Over $[0, 2\pi]$, how many values of $x$ can be found satisfying this equation?

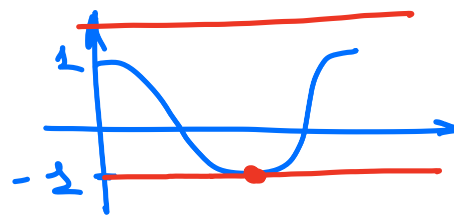
a) $\sin(x) = \frac{1}{2}$ $\boxed{2}$

sin x




b) $\cos(x) = -1$ $\boxed{1}$

cos x




c) $4\cos(x) = 5$ $\boxed{0}$

$\cos(x) = \frac{5}{4}$

d) $6\sin(4x) + 1 = 5$

$6\sin(4x) = 4$

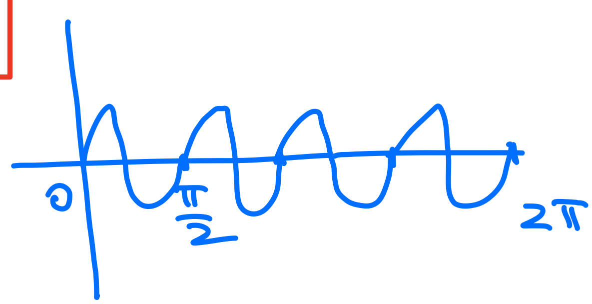
$\sin(4x) = \frac{4}{6}$

Period $= \frac{2\pi}{B} = \frac{2\pi}{4} = \frac{\pi}{2}$

$\sin(4x) = \frac{2}{3}$

$\sin(x) = \frac{2}{3}$ 2 sol.

$\boxed{\text{8 solutions}}$



e) $4\cos^2(2x) - 1 = 1$

$4\cos^2(2x) = 2$

$\sqrt{\cos^2(2x)} = \pm\sqrt{\frac{1}{2}}$

$\cos(2x) = +\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\qquad$ $\cos(2x) = -\frac{\sqrt{2}}{2}$

$[0, 2\pi]$

$2 \times 2 = 4$ $\qquad$ $\underline{\underline{4}}$

8 solutions.