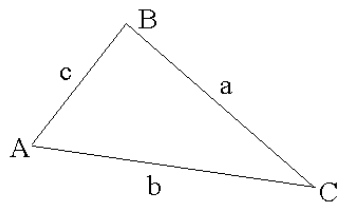
Section 7.3
Law of Sines and Law of Cosines

We use these laws to find angles and side lengths for triangles of any type (not just right triangles).

Law of Sines

Given a triangle labeled as:

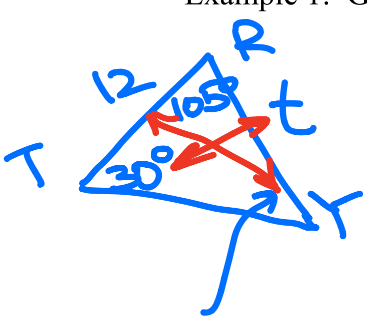


The sines of angles are proportional to the lengths of opposite sides.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The law of sines is used for the following two cases: SAA – One side and two angles, SSA – Two sides and an angle opposite one of the given sides.

Example 1: Given ΔTRY, y = 12 cm, $\angle T = 30^o$ and $\angle R = 105^o$. Find t.



$180 - 105 - 30$
$= 45^\circ$

$$\frac{\sin 30^\circ}{t} = \frac{\sin 45^\circ}{12}$$

$$t \sin 45^\circ = 12 \sin 30^\circ$$

$$t \frac{\sqrt{2}}{2} = 12 \cdot \frac{1}{2}$$

$$\frac{\sqrt{2}}{2} t = 6 \qquad t = \frac{6}{\frac{\sqrt{2}}{2}} = 6 \cdot \frac{2}{\sqrt{2}}$$

$$= \frac{12}{\sqrt{2}} = \frac{12\sqrt{2}}{2} = \boxed{6\sqrt{2} \text{ cm}}$$

Note: *SSA* case is called the ambiguous case of the law of sines. There may be two solutions, one solution, or no solutions. You should throw out the results that don't make sense. That is, if $\sin A > 1$, $\sin A < -1$ or the angles add up to more than 180°.

Example 2: In triangle ABC, angle A = 45°, a = 8 m and b = $4\sqrt{2}$ m. Find all possible measures for angle B.

$$\frac{\sin 45^\circ}{8} = \frac{\sin B}{4\sqrt{2}}$$

$$4\sqrt{2} \cdot \sin 45^\circ = 8 \sin B$$

$$4\sqrt{2} \cdot \frac{\sqrt{2}}{2} = 8 \sin B$$

$$\frac{8}{2} = 8 \sin B$$

$$\sin B = \frac{1}{2}$$

Possible solutions:

$\boxed{m\angle B = 30^\circ}$ ~~$m\angle B = 150^\circ$~~

Check:

$45 + 30 < 180^\circ$

$45 + 150 = 195 > 180^\circ$

Example 3: For triangle ABC, with $a = 2\text{ft}$, $b = 10\text{ft}$, $A = 30^\circ$, find the length of the other side and
the measure of the remaining angles. $\angle B$, $\angle C$ c

$$\frac{\sin 30^\circ}{2} = \frac{\sin B}{10}$$

$$2 \sin B = 10 \sin 30^\circ$$

$$2 \sin B = 10 \cdot \frac{1}{2}$$

$$2 \sin B = 5$$

$$\sin B = \frac{5}{2} > 1$$

NOT possible

NO solution

Example 4: Find all possible measures for the indicated angle of the triangle.

ΔDEF

d = 25 mm

e = 13 mm

$\angle E = 21^o$

Find $\angle D$.

$$25 \cdot \frac{\sin 21^\circ}{13} = \frac{\sin D}{\cancel{25}} \cdot \cancel{25}$$

$$\sin D = \frac{25}{13} \sin(21^\circ)$$

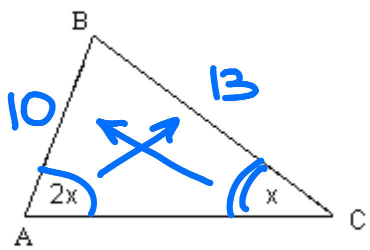
$$m\angle D = \arcsin\left(\frac{25}{13}\sin(21^\circ)\right) \overset{\text{calc.}}{\approx} 43.56^\circ \text{ OK}$$

$$m\angle D = 180 - \arcsin\left(\frac{25}{13}\sin(21^\circ)\right) \approx 136.44^\circ \text{ OK}$$

Check: $21 + 136.44 < 180$

Two solutions!

Try this one: Determine the angle x in the triangle given below with AB = 10cm and BC = 13cm.



$$\frac{\sin x}{10} = \frac{\sin(2x)}{13}$$

$$10\sin(2x) = 13\sin(x)$$

$$10\sin(2x) - 13\sin(x) = 0$$

$$\sin(2x) = 2\sin x\cos x$$

$$10\cdot 2\sin(x)\cos(x) - 13\sin(x) = 0$$

$$20\sin(x)\cos(x) - 13\sin(x) = 0$$

$$\sin(x)[20\cos(x) - 13] = 0$$

$\sin(x) = 0$ or $20\cos(x) - 13 = 0$

$x = 0°$ or $x = 180°$ NOT possible

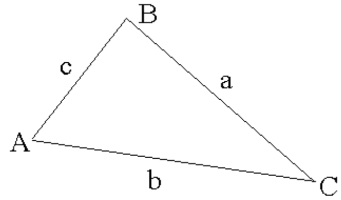
$$20\cos(x) = 13$$

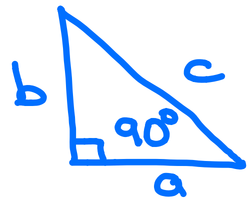
$$\cos(x) = \frac{13}{20}$$

$$\boxed{x = \arccos\left(\frac{13}{20}\right) \approx 49.46°}$$

Law of Cosines:

Given a triangle labeled as:





$c^2 = a^2 + b^2 - \cancel{2ab\cos 90^\circ}$ 0

We have the generalized Pythagorean Theorem:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

The law of cosines is used for the following two cases: *SAS* – Two sides and the included angle, *SSS* – three sides are given.

For this law, you'll have no solution if $\cos A > 1$ or $\cos A < -1$.

Example 5: Given ΔPEZ, p = 6 cm, e = 13 cm, and z = 11 cm. Find $\angle$Z.

$$11^2 = 6^2 + 13^2 - 2(6)(13)\cos Z$$

$$121 = 36 + 169 - 156\cos Z$$

$$121 - 36 - 169 = -156\cos Z$$

$$-84 = -156\cos Z$$

$$\cos Z = \frac{-84}{-156} = \frac{84}{156} = \frac{42}{78} = \frac{21}{39} = \frac{7}{13}$$

$$\cos Z = \frac{7}{13} \qquad \angle Z = \cos^{-1}\left(\frac{7}{13}\right) \approx 57.42^\circ$$

SSS

Example 6: Given ΔRUN, $r = 7$ cm, $u = 12$ cm, and $n = 4$ cm. Find the measure of angle U.

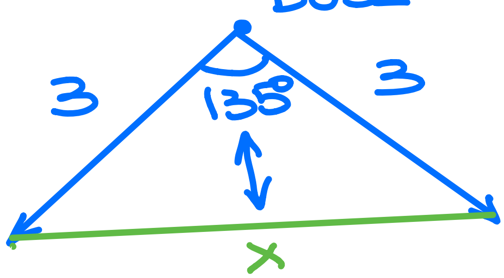
$$12^2 = 7^2 + 4^2 - 2(7)(4)\cos U$$
$$144 = 49 + 16 - 56\cos U$$
$$144 - 49 - 16 = -56\cos U$$
$$79 = -56\cos U$$
$$\cos U = -\frac{79}{56} < -1$$

$7 + 4 < 12$

$11 < 12$

NO solution

Example 7: Two sailboats leave the same dock together traveling on courses that have an angle of 135° between them. If each sailboat has traveled 3 miles, how far apart are the sailboats from each other?



SAS

$$x^2 = 3^2 + 3^2 - 2(3)(3)\cos(135°)$$
$$x^2 = 9 + 9 - \cancel{18}^{\,9}\left(-\frac{\sqrt{2}}{\cancel{2}}\right)$$
$$\sqrt{x^2} = \sqrt{18 + 9\sqrt{2}}$$
$$x = \sqrt{18 + 9\sqrt{2}} = \sqrt{9(2+\sqrt{2})} = \sqrt{9}\sqrt{2+\sqrt{2}}$$
$$= \boxed{3\sqrt{2+\sqrt{2}}} \text{ miles}$$