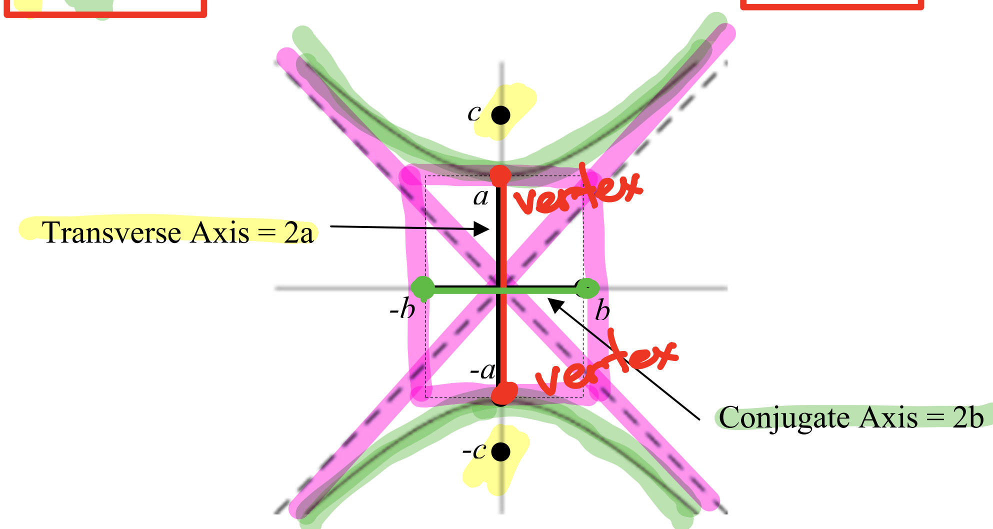
Section 8.3
Hyperbolas

elipse sum

A **hyperbola** is the set of all points, the difference of whose distances from two fixed points is constant. Each fixed point is called a **focus** (plural = *foci*). The **focal axis** is the line passing through the foci.

Basic "Vertical" Hyperbola (center is at the origin):

Equation: $\boxed{\dfrac{y^2}{a^2}-\dfrac{x^2}{b^2}=1}$ Foci Formula: $\boxed{c^2=a^2+b^2}$



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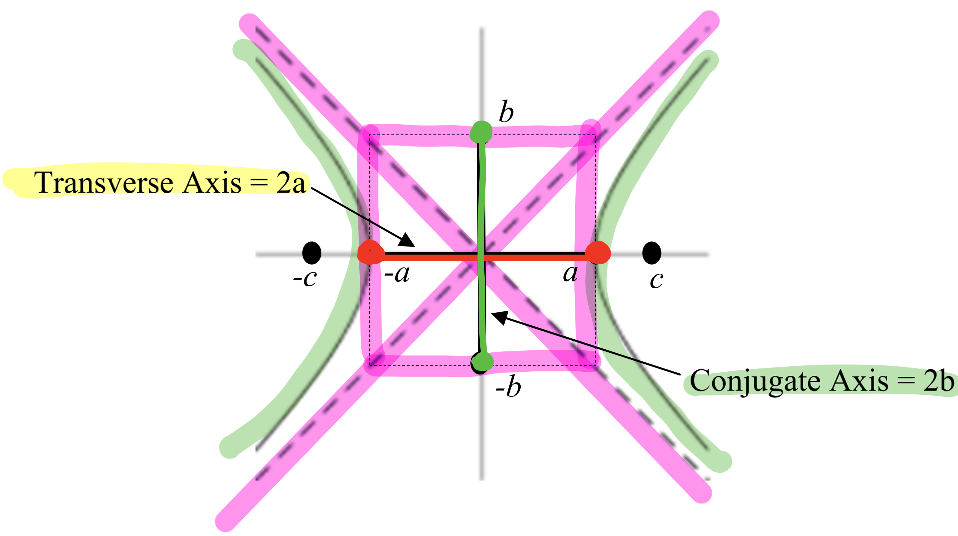


To graph a hyperbola with center at the origin:

- Rearrange the equation into the form $\boxed{\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \text{ or } \dfrac{y^2}{a^2}-\dfrac{x^2}{b^2}=1.}$

- If the x^2 is positive, it's horizontal. If the y^2 is positive, it's vertical.

- Use the square root of the number under x^2 to determine how far to measure in x-direction from the center.

- Use the square root of the number under y^2 to determine how far to measure in y-direction from the center.

- Draw a rectangle with these measurements. Then draw two diagonals through the center, from one corner of the rectangle to the opposite corner. These are the asymptotes.

- Use the dimensions of the rectangle to determine the slopes of the asymptotes and write their equations.

- Put the vertices at the edge of the box on the correct axis. Then draw a hyperbola, making sure it approaches the asymptotes smoothly.

- Find the foci. Formula: $c^2=a^2+b^2$ where a^2 and b^2 are the denominators. The foci are located c units from the center, on the same axis as the vertices.

For hyperbolas:
In the formulas at the top, "a" is associated with the POSITIVE TERM!

Example 1: Given the hyperbola in standard form $\frac{x^2}{36} - \frac{y^2}{4} = 1$, sketch its graph and state each of the following features listed below.

Orientation: Horizontal

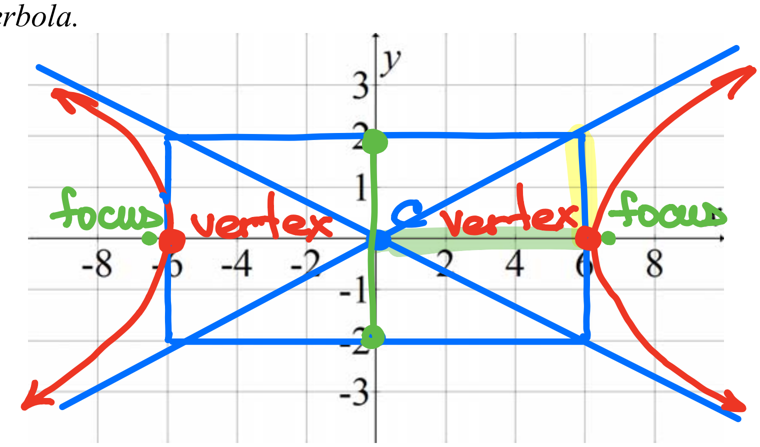
Center: $(0,0)$

Recall: For hyperbolas, "a" is associated with the positive term.

x-direction: $\sqrt{36} = 6 = a$

y-direction: $\sqrt{4} = 2 = b$

Next, draw the hyperbola.



$m = \frac{\text{rise}}{\text{run}}$

Asymptotes:

$m = \frac{2}{6} = \frac{1}{3}$ $\quad y = \pm m x \quad y = \pm \frac{1}{3} x$

Vertices: $(6,0)$ & $(-6,0)$

Foci Formula: $c^2 = a^2 + b^2 = 36 + 4 = 40$

$c = 2\sqrt{10}$

$36 < 40 < 49 \quad 6 < \sqrt{40} < 7$

Foci: $(2\sqrt{10}, 0)$ & $(-2\sqrt{10}, 0)$

Length of the Transverse Axis: $2a = 2(6) = 12$

Coordinates of the Transverse Axis: $(6,0)$ & $(-6,0)$

Length of the Conjugate Axis: $2b = 2(2) = 4$

Coordinates of the Conjugate Axis: $(0,2)$ & $(0,-2)$

To graph a hyperbola with center not at the origin:

- Rearrange the equation into the form $\boxed{\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1 \text{ or } \dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1}$
- Start at the center (h,k) and then graph it as before.
- To write down the equations of the asymptotes, start with the equations $y = \pm mx$ and replace x with $x-h$ and replace y with $y-k$.

Example 2: Write $y^2 - 25x^2 + 8y - 9 = 0$ in standard form. Sketch its graph and state each of the following features listed below.

$$(y^2 + 8y + 16) - 25x^2 = 9 + 16 \quad \Big| \left(\frac{8}{2}\right)^2 = (4)^2 = 16$$

$$\frac{(y+4)^2}{25} - \frac{25x^2}{25} = \frac{25}{25}$$

$$\frac{(y+4)^2}{25} - \frac{x^2}{1} = 1$$

Orientation: vertical

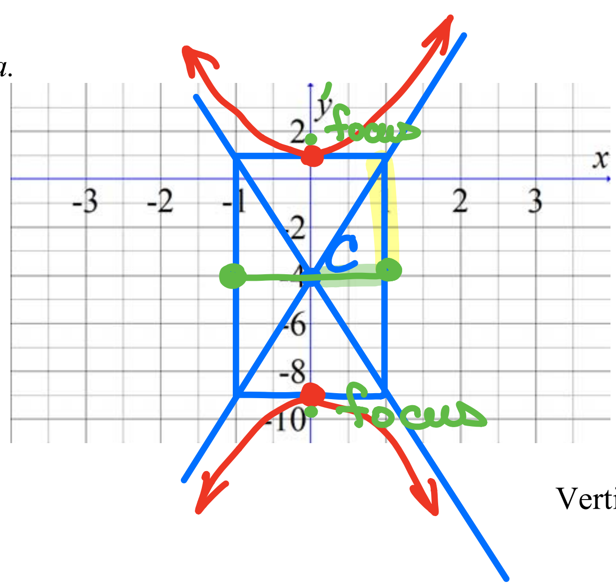
Center: $(0,-4)$

Recall: For hyperbolas, "a" is associated with the positive term.

x-direction: $\sqrt{1} = 1 = b$

y-direction: $\sqrt{25} = 5 = a$

Next, draw the hyperbola.



Asymptotes:

$y = \pm \frac{5}{1} x$

Vertices: $(0,1)$ & $(0,-9)$

$y + 4 = \pm 5x \qquad y = \pm 5x - 4$

Foci Formula: $c^2 = a^2 + b^2$ $= 25 + 1 = 26$

$c = \sqrt{26}$

$25 < 26 < 36$ $5 < \sqrt{26} < 6$

Foci: $(0, -4 + \sqrt{26})$ & $(0, -4 - \sqrt{26})$

Length of the Transverse Axis:

$2(5) = 10$

Coordinates of the Transverse Axis:

$(0, 1)$ & $(0, -9)$

Length of the Conjugate Axis:

$2(1) = 2$

Coordinates of the Conjugate Axis:

$(1, -4)$ & $(-1, -4)$

Example 3: Use the given features to answer the following questions.
Center: (6, 8) $= (h, k)$
a ← Length of transverse axis = 16 $\Rightarrow$ $a = 8$
b ← Length of conjugate axis = 6 $\Rightarrow$ $b = 3$
Horizontal transverse axis

a. Which of the following equations will model the hyperbola?

A. $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ B. $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

b. Write an equation for the hyperbola in standard form.

$$\frac{(x-6)^2}{64} - \frac{(y-8)^2}{9} = 1$$