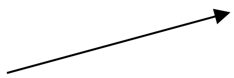
Math 2312
Vectors

Quantities that involve BOTH a magnitude and a direction are called **vectors**.

Quantities that involve magnitude, but no direction are called **scalars**.

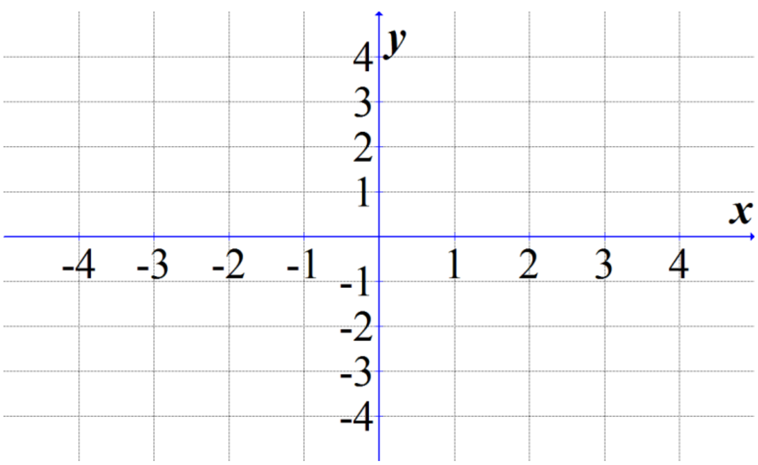
Example: You are driving south at 55 miles per hour. The magnitude is the speed of your car and the direction is south.

Geometrically, a vector is a directed line segment, i.e. a line segment that has an **initial point** and a **terminal point**.



Magnitude is the length of a vector $\vec{u}$ and given by $\| \vec{u} \| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, where (x_1, y_1) and (x_2, y_2) are the initial and the terminal points of a vector.

Example: Let vector $\vec{u}$ be a vector with initial point $(-4, -1)$ and a terminal point $(-1, 2)$, find the magnitude of this vector.



Unit Vectors and Representing Vectors in Rectangular Coordinates

Vector $\vec{\imath}$ is the unit vector whose direction is along the positive x-axis.
Vector $\vec{\jmath}$ is the unit vector whose direction is along the positive y-axis.
Vectors in the rectangular coordinate system can be represented in terms of $\vec{\imath}$ and $\vec{\jmath}$.

Vector $\vec{v}$, from $(0,0)$ to (a, b), is represented as $\vec{v} = a\vec{\imath} + b\vec{\jmath}$, where

- a is the horizontal component of $\vec{v}$, and
- b is the vertical component of $\vec{v}$.

The magnitude of $\vec{v} = a\vec{\imath} + b\vec{\jmath}$ is equal to $\sqrt{a^2 + b^2}$.

Example: Find the magnitude of a vector $\vec{u} = 2\vec{\imath} - 3\vec{\jmath}$.

Vector $\vec{v}$ with initial point $P_1 = (x_1, y_1)$ and terminal point $P_2 = (x_2, y_2)$ is equal to the position vector $\vec{v} = (x_2 - x_1)\vec{\imath} + (y_2 - y_1)\vec{\jmath}$.

Example: Let $\vec{v}$ be a vector from initial point $(-8,6)$ to terminal point $(-2, 3)$. Write $\vec{v}$ in terms of $\vec{\imath}$ and $\vec{\jmath}$.

Example: Let $\vec{v}$ be a vector from initial point $(-3,4)$ to terminal point $(6, 4)$. Find the vertical and the horizontal component of this vector.

Vertical component: Horizontal component:

Operations with Vectors

If vectors are expressed in terms of $\vec{i}$ and $\vec{j}$, we can easily carry out operations such as vector addition, vector subtraction, and scalar multiplication.

If $\vec{u} = a\vec{i} + b\vec{j}$ and $\vec{v} = c\vec{i} + d\vec{j}$, then

- $\vec{u} + \vec{v} = (a + c)\vec{i} + (b + d)\vec{j}$
- $\vec{u} - \vec{v} = (a - c)\vec{i} + (b - d)\vec{j}$
- $k\vec{u} = (ka)\vec{i} + (kb)\vec{j}$

  If k is a real number and $\vec{u}$, the vector $k\vec{u}$ is called a scalar multiple of the vector $\vec{u}$.
  Multiplying a vector by any positive real number other than 1 changes its magnitude, but not its direction.
  Multiplying a vector by any negative real number other than -1 changes its magnitude, AND its direction.
  The magnitude of $k\vec{u}$ is $|k| \parallel \vec{u} \parallel$.

Example: If $\vec{u} = 2\vec{i} + 3\vec{j}$ and $\vec{v} = -\vec{i} - 4\vec{j}$, find

a. $\vec{u} + \vec{v}$

b. $\vec{u} - \vec{v}$

c. $3\vec{u} - 2\vec{v}$

Unit Vectors

A **unit vector** is a vector whose magnitude is one.

In many applications it is useful to find the unit vector that has the same direction as a given vector.

For any nonzero vector $\vec{v}$, the vector $\frac{\vec{v}}{\|\vec{v}\|}$ is the unit vector that has the same direction as $\vec{v}$.

To find this vector, divide $\vec{v}$ by its magnitude.

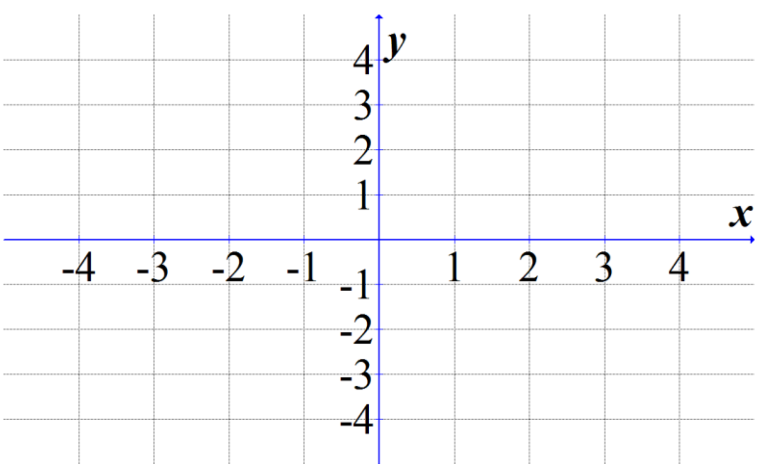
Example: Find the unit vector in the same direction as $\vec{v} = 4\vec{i} + 3\vec{j}$.

A Vector in Terms of Its Magnitude and Direction

The components a and b of the vector $\vec{v} = a\vec{i} + b\vec{j}$ can be expressed in terms of the magnitude of $\vec{v}$ and the angle θ it makes with the positive x-axis.

$$\vec{v} = a\vec{i} + b\vec{j} = \| \vec{v} \| \cos\theta\, \vec{i} + \| \vec{v} \| \sin\theta\, \vec{j}$$

θ is called the **direction angle**.



Example: Write the vector $\vec{v}$ in terms of $\vec{\imath}$ and $\vec{\jmath}$ if its magnitude and direction angle are given.
$\| \vec{v} \| = 5, \theta = 60^{\circ}$.

Example: Given the vector $\vec{v} = -3\vec{\imath} + 3\sqrt{3}\vec{\jmath}$, find the direction angle.

The Dot Product of Two Vectors

If $\vec{u} = a\vec{\imath} + b\vec{\jmath}$ and $\vec{v} = c\vec{\imath} + d\vec{\jmath}$, then the dot product of $\vec{u} \cdot \vec{v} = ac + bd$.

The dot product of two vectors is the sum of the products of their horizontal components and vertical components.

The result of vector addition and scalar multiplication is a vector.
The result of the dot product is a SCALAR (a real number).

Example: If $\vec{u} = 2\vec{\imath} + 3\vec{\jmath}$ and $\vec{v} = -\vec{\imath} - 4\vec{\jmath}$, find $\vec{u} \cdot \vec{v}$.

Properties of Dot Product

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{0} \cdot \vec{v} = 0$
3. $\vec{v} \cdot \vec{v} = \| \vec{v} \|^2$
4. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
5. $(\mathrm{c}\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$

Alternative Formula for the Dot Product: $\vec{u} \cdot \vec{v} = \| \vec{u} \| \| \vec{v} \| \cos\theta$, where θ is the smallest non-negative angle between vectors $\vec{u}$ and $\vec{v}$.

Formula for the Angle between Two Vectors: $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$ and $\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)$.

Example: If $\vec{u} = -3\vec{\imath} + 4\vec{\jmath}$ and $\vec{v} = 5\vec{\imath} + 12\vec{\jmath}$, find the angle between $\vec{u}$ and $\vec{v}$.

Parallel and Orthogonal Vectors

Two vectors are **parallel** when the angle θ between the vectors is 0° or 180°.

If $\theta = 0^\circ$, the vectors point in the same direction.

If $\theta = 180^\circ$, the vectors point in opposite directions.

Two vectors are orthogonal when the angle between them is 90°.

Two nonzero vectors are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$.

Example: Are the vectors $\vec{u} = 2\vec{\imath} + 3\vec{\jmath}$ and $\vec{v} = 12\vec{\imath} - 8\vec{\jmath}$ orthogonal?

Example: Determine if $\vec{u} = 3\vec{\imath} - 5\vec{\jmath}$ and $\vec{v} = 6\vec{\imath} - 10\vec{\jmath}$ are parallel, orthogonal, or neither.

Example: Given vectors $\langle 2,3 \rangle$ and $\langle -4, x \rangle$ are orthogonal, determine the value of x.