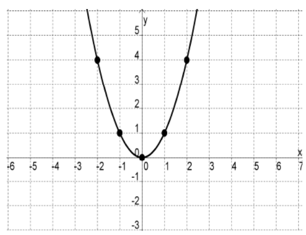
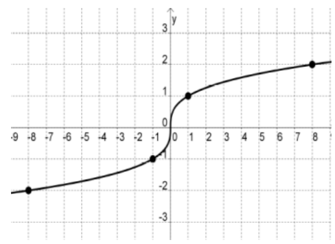
Section 4.1 – Inverse Functions

A function is said to be **one-to-one** (1-1) if there are no two distinct numbers in the domain of f that produce the same value.

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

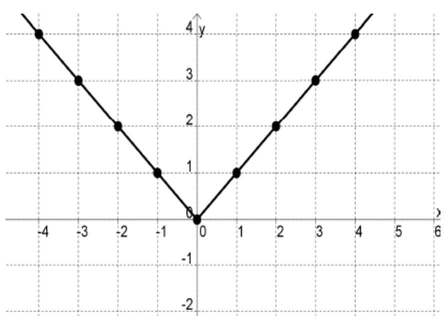
In other words, two different x values cannot have the same y value.

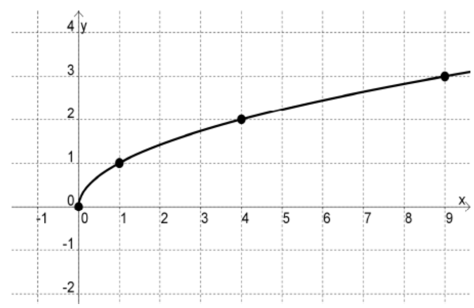


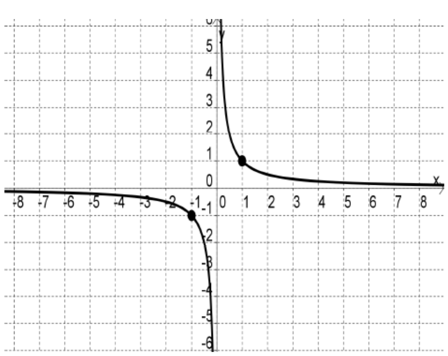


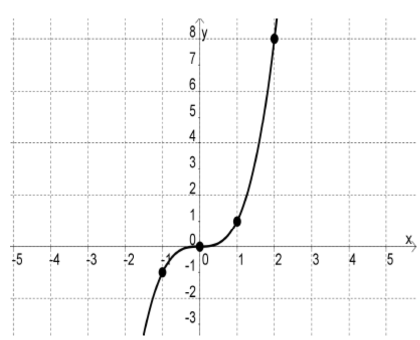
Given a function whose graph is known or given the graph of a function, we can use the Horizontal Line Test to determine if the function is 1-1.

Example 1: Which of these functions are one-to-one?









Using the definition to prove a function is 1-1 or not

A function is said to be **one-to-one** (1-1) if there are no two distinct numbers in the domain of f that produce the same value.

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

Example 2: Prove that $f(x) = (x + x^2)^7$ is *not* one-to-one?

Example 3: Prove that $f(x) = \sqrt[3]{4x - 3} + 2$ is 1-1.

If a function is 1-1, then it has an inverse function, denoted as f^{-1}, which reverses what the first function did.

Definition: Let f be a one-to-one function. There exists a unique function f^{-1}, called the inverse of f, such that for each x in the domain of f:

$$f^{-1}(f(x)) = x.$$

The domain of f is the range of $f^{-1}$and the range of f is the domain of f^{-1}.

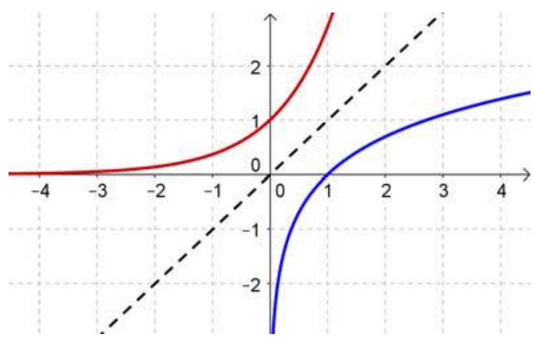
How are functions related to their inverses?

Algebraically

$$\text{If } f(a) = b, \text{ then } f^{-1}(b) = a$$

If (a, b) is a point on the graph of $f(x)$, then (b, a) is a point on the graph of $f^{-1}(x)$.

Geometrically



How to check if two functions are inverses of each other?

Let f and g be two functions such that $(f \circ g)(x) = x$ for every x in the domain of g and $(g \circ f)(x) = x$ for every x in the domain of f, then ***f* and *g* are inverses of each other.**

An example of real life inverse function is:
The formula $f(x) = \frac{9}{5}x + 32$ is used to convert from x degrees Celsius to y degrees Fahrenheit.
The formula $g(x) = \frac{5}{9}(x - 32)$ is used to convert from x degrees Fahrenheit to y degrees Celsius.

Verify $f(x) = \frac{9}{5}x + 32$ and $g(x) = \frac{5}{9}(x - 32)$ are inverses of each other.

So we need to check to see if: $(f \circ g)(x) = x$ AND $(g \circ f)(x) = x$.

1. $(f \circ g)(x)$

2. $(g \circ f)(x)$

How do we find the formula for the inverse of a function?

1. Replace $f(x)$ by y.
2. Exchange x and y.
3. Solve for y.
4. Replace y by $f^{-1}(x)$.

Example 3 (cont'd): Find the equation of the inverse for $f(x) = \sqrt[3]{4x-3} + 2$.

Sometimes it is too long or too difficult to find the equation of the inverse, yet we may want to know if a function has an inverse or not.

Definition: A function is **monotonic** if it is always increasing or always decreasing on its domain.

Recall:

- If $f'(x) > 0$ on its domain, then f is increasing and; hence, monotonic.
- If $f'(x) < 0$ on its domain, then f is decreasing and; hence, monotonic.

Theorem: If f is monotonic, then f is an invertible function.

Example 4: Is the following function 1-1?

a. $f(x) = x^3 + 3x$

b. $f(x) = \frac{x-1}{x+1}$

Example 5: Let $f(x) = \frac{1}{3}x^3 - x^2 + kx$. For what values of k is $f(x)$ invertible?

Finding the derivative of the inverse function

Theorem: If $f(x)$ is continuous and invertible then $f^{-1}(x)$ is continuous.

Theorem: If $f(x)$ is differentiable (so must be continuous) and invertible, and$f^{-1}(x) \neq 0$, then $f^{-1}(x)$ is differentiable.

If $f(a) = b$ and $f'(a) \neq 0$, then $(f^{-1})'(b) = \frac{1}{f'(a)}$.

Example 6: For $f(x) = x^3$, we know that $f(2) = 8$. Find $(f^{-1})'(8)$.

Example 7: If f is invertible and $f(1) = 2,\ f(3) = 1,\ f'(1) = 4,\ f'(3) = 5,\ f'(2) = 6$, find $(f^{-1})'(1)$.

Example 8: Given $f(x) = x^5 + 1$, find $(f^{-1})'(33)$if possible.

Example 9: If $f(x) = \sin x + 5\cos x$, $x \in \left[0, \frac{\pi}{2}\right]$, find $(f^{-1})'\left(3\sqrt{2}\right)$.

Example 10: Let $f(x) = x^5 + 2x^3 + 2x$. The point $(-5, -1)$ is on the graph of $f^{-1}(x)$. Find

$(f^{-1})'(-5)$, then give an equation for the tangent line to the graph of $f^{-1}(x)$ at the point $(-5, -1)$.