Section 4.5 – Hyperbolic Functions

We will now look at six special functions, which are defined using the exponential functions $e^x$ and $e^{-x}$. These functions have similar names, identities, and differentiation properties as the trigonometric functions. While the trigonometric functions are closely related to circles, the hyperbolic functions earn their names due to their relationship with hyperbolas. The first two functions we will define are the hyperbolic sine and hyperbolic cosine functions.

**Hyperbolic Cosine:** \( \cosh(x) = \frac{e^x + e^{-x}}{2} \)

**Hyperbolic Sine:** \( \sinh(x) = \frac{e^x - e^{-x}}{2} \)

The four remaining hyperbolic functions are defined, as you would expect given their names. That is:

\[
\begin{align*}
\tanh(x) &= \frac{\sinh(x)}{\cosh(x)} \\
\coth(x) &= \frac{\cosh(x)}{\sinh(x)} \\
\text{sech}(x) &= \frac{1}{\cosh(x)} \\
\text{csch}(x) &= \frac{1}{\sinh(x)}
\end{align*}
\]

Let us examine the graphs of these two new functions. Below we have the graph of the hyperbolic sine function, as well as the two exponential functions used to define it.

Hyperbolic sine function is an ODD function, i.e. \( \sinh(-x) = -\sinh x \).

Domain: \((-\infty, \infty)\); Range: \((-\infty, \infty)\)
Next, we will look at the graph of the hyperbolic cosine function below.

![Graph of the hyperbolic cosine function](image)

Hyperbolic cosine function is an EVEN function, i.e. $\cosh(-x) = \cosh x$.

Domain: $(-\infty, \infty)$; Range $[1, \infty)$

**Some important properties:**

\[
\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} =
\]

\[
\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} =
\]

Similarly, we can prove that $\cosh^2 x - \sinh^2 x = 1$.

We should notice this identity is similar to the Pythagorean trigonometric identity satisfied by the sine and cosine functions, namely $\cos^2 x + \sin^2 x = 1$. In addition, we can see how these functions earn the hyperbolic distinction by setting $x = \cosh(t)$ and $y = \sinh(t)$ and substituting into $x^2 - y^2 = 1$, which is the equation of a hyperbola.
IMPORTANT IDENTITIES:

\[ \cosh^2 x - \sinh^2 x = 1 \]
\[ \cosh x + \sinh x = e^x \]
\[ \cosh x - \sinh x = e^{-x} \]

Example 1: Given \( f(x) = \cosh(2x) \), find each of the following function values.

Recall: \( \cosh(x) = \frac{e^x + e^{-x}}{2} \)

a. \( f(1) \)

b. \( f(\ln(3)) \)
Derivatives of the Hyperbolic Functions

Example 2: Differentiate \( \sinh(x) = \frac{e^x - e^{-x}}{2} \).

Derivatives of six basic hyperbolic functions:

\[
\begin{align*}
\frac{d}{dx} \sinh x &= \cosh x \\
\frac{d}{dx} \tanh x &= \text{sech}^2 x \\
\frac{d}{dx} \coth x &= -\text{csch}^2 x
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dx} \cosh x &= \sinh x \\
\frac{d}{dx} \text{sech} x &= -\tanh x \text{sech} x \\
\frac{d}{dx} \text{csch} x &= -\coth x \text{csch} x
\end{align*}
\]

The six differentiation formulas are in the table below, written in chain rule form.

\[
\begin{align*}
\frac{d}{dx} (\sinh u) &= \cosh(u) \frac{du}{dx} \\
\frac{d}{dx} (\tanh u) &= \text{sech}^2(u) \frac{du}{dx} \\
\frac{d}{dx} (\coth u) &= -\text{csch}^2(u) \frac{du}{dx}
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dx} (\cosh u) &= \sinh(u) \frac{du}{dx} \\
\frac{d}{dx} (\text{sech} u) &= -\tanh(u) \text{sech}(u) \frac{du}{dx} \\
\frac{d}{dx} (\text{csch} u) &= -\coth(u) \text{csch}(u) \frac{du}{dx}
\end{align*}
\]
**Example 3:** Differentiate: \( y = \sinh \sqrt{\cos x} \).

**Example 4:** Differentiate: \( y = \arctan(\sinh(11x)) \).
Example 5: Find \( \frac{d}{dx} \left[ \frac{\cosh x}{1+\sinh x} \right] \).

Example 6: Differentiate: \( f(x) = \cosh(\ln(4x^3)) \).
Example 7: Differentiate: $y = (\sinh x)^x$. 