Section 5.3 – L'Hospital's Rule

Recall that earlier we first saw the indeterminate form of $0/0$. We were able to evaluate them further by either factoring, rationalizing, graphing the functions, etc, but that sometimes is not enough to calculate the limit. Where these elementary methods are difficult to apply, we may be able to use L'Hospital's rule. This rule makes the calculation much easier.

L'Hospital's Rule

Let f , g be two functions differentiable on an open interval I that contains c and suppose that $g'(x) \neq 0$ on I (except possibly at c). If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, then

$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$, provided that the limit on the right hand side exists.

On the other hand, $\frac{f'(x)}{g'(x)} \to \pm\infty$ as $\to c$, then $\frac{f(x)}{g(x)} \to \pm\infty$ as well.

This result also applies when the limit of $f(x)/g(x)$ as x approaches c produces one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

Note: You are NOT using the quotient rule here!

In some cases, applying L'Hospital's rule once still produces an indeterminate form. We can apply the rule several times as long as the conditions of the rule are met. As soon as we get a limit that is not indeterminate, we should stop.

Indeterminate Form 0/0

Example 1: Evaluate each limit, if it exists.

a. $\lim_{x\to 4} \frac{x-4}{x^4-256} = \frac{4-4}{4^4-256} = \frac{0}{0}$

$$\lim_{x\to 4} \frac{x-4}{x^4-256} = \lim_{x\to 4} \frac{1}{4x^3} = \frac{1}{4\cdot 4^3} = \frac{1}{256}$$

b. $\lim\limits_{x\to 0}\frac{10\ \arctan x}{x} = \frac{10\arctan 0}{0} = \frac{0}{0}$

$$\lim_{x\to 0}\frac{10\arctan x}{x} = \lim_{x\to 0}\frac{10\cdot\frac{1}{1+x^2}}{1} = 10\cdot\frac{1}{1+0^2} = 10$$

c. $\lim\limits_{x\to \pi}\frac{1+\cos x}{x-\pi} = \frac{1+\cos\pi}{\pi-\pi} = \frac{1-1}{0} = \frac{0}{0}$

$$\lim_{x\to \pi}\frac{1+\cos x}{x-\pi} = \lim_{x\to \pi}\frac{-\sin x}{1} = -\sin(\pi) = 0$$

d. $\lim\limits_{x\to 0}\frac{5e^{3x}-5}{x^2} = \frac{5e^0-5}{0^2} = \frac{0}{0}$

$$\lim_{x\to 0}\frac{5e^{3x}-5}{x^2} = \lim_{x\to 0}\frac{5\cdot 3\cdot e^{3x}}{2x} = \frac{15e^0}{2(0)} = \frac{15}{0}\ \text{DNE}\ (\infty)$$

Indeterminate Form ∞/∞

Example 2: Evaluate each limit, if it exists.

a. $\lim_{x\to\infty} \frac{5\sqrt{1+x^2}}{6x^2} = \frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{5\sqrt{1+x^2}}{6x^2} = \lim_{x\to\infty} \frac{5\cdot\frac{1}{2}(1+x^2)^{-1/2}\cdot 2x}{12x}$$

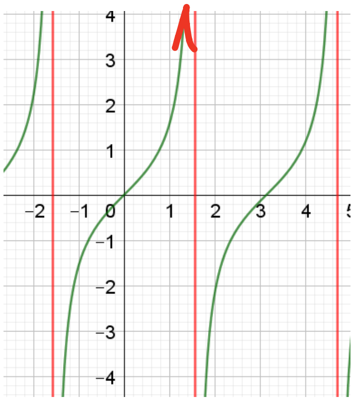
$$= \lim_{x\to\infty} \frac{5}{12\,(1+x^2)^{1/2}} = 0$$

b. $\lim_{x\to\infty} \frac{\ln^2 x}{x} = \frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{\ln^2 x}{x} = \lim_{x\to\infty} \frac{2\cdot\ln x\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{\overset{\nearrow\infty}{2\ln x}}{\underset{\searrow\infty}{x}}$$

$$= \lim_{x\to\infty} \frac{2\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2}{x} = 0$$

c. $\lim_{x\to\frac{\pi}{2}^-} \frac{\sec x}{1+\tan x} = \frac{\sec\left(\frac{\pi}{2}\right)}{1+\tan\left(\frac{\pi}{2}\right)} = \frac{\infty}{\infty}$



$$\lim_{x\to\frac{\pi}{2}^-} \frac{\sec x}{1+\tan x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\cancel{\sec x}\tan x}{\sec^{\cancel{2}} x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\overset{\nearrow\infty}{\tan x}}{\underset{\searrow\infty}{\sec x}}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{\sin x}{\cos x}\cdot\frac{\cos x}{1} = \lim_{x\to\frac{\pi}{2}^-} \sin x = 1$$

Other indeterminate forms exist. Let's see what we do…

Indeterminate Form: $\mathbf{0} \cdot \infty$ ⇒ Rewrite the function so that you get $\mathbf{0/0}$ or ∞/∞.

Generally in the case of $\mathbf{0} \cdot \infty$, you'll need to rewrite the expression $A \cdot B$ as $\frac{A}{\frac{1}{B}}$ or $\frac{B}{\frac{1}{A}}$ so that it either becomes $0/0$ or ∞/∞ and then you'll be able to use L'Hospital's rule. Deciding on which form to rewrite it into, remember you will need to take their derivatives, reciprocate the one that is easier to differentiate.

Example 3: Evaluate the limit, if it exists. $\lim\limits_{x\to\infty} \overset{\infty \cdot 0}{\left(x^2 \sin\frac{1}{x}\right)}$

$$\lim_{x\to\infty}\left(x^2\sin\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\sin\frac{1}{x} \rightarrow 0}{\frac{1}{x^2} \rightarrow 0} = \lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{x^{-2}}$$

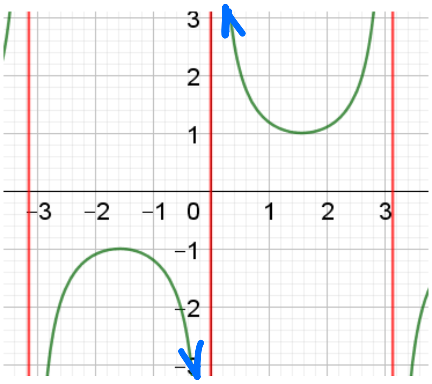
$$= \lim_{x\to\infty}\frac{\cos\left(\frac{1}{x}\right)\left(-x^{-2}\right)}{-2x^{-3}} = \lim_{x\to\infty}\frac{\cos\left(\frac{1}{x}\right)x^{\cancel{3}}}{\cancel{x^2}\cdot 2}$$

$$= \lim_{x\to\infty}\frac{\cos\frac{1}{x} \rightarrow 1}{\frac{2}{x} \rightarrow 0} = \text{DNE}$$

$$(\infty)$$

Indeterminate Form: $\infty - \infty \Rightarrow$ **Rewrite the function so that you get $0/0$ or ∞/∞.**

Example 4: $\lim_{x\to 0^+}\left(\frac{1}{x} - \csc x\right) = \infty - \infty$



$$\lim_{x\to 0^+}\left(\frac{1}{x} - \csc x\right) = \lim_{x\to 0^+}\left(\frac{1}{x}\overset{\sin x}{} - \frac{1}{\sin x}\overset{x/}{}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{\sin x - x \xrightarrow{} 0}{x\sin x \xrightarrow{} 0}\right) = \lim_{x\to 0^+}\frac{\cos x - 1 \xrightarrow{} 0}{\sin x + x\cos x \xrightarrow{} 0}$$

$$= \lim_{x\to 0^+}\frac{-\sin x \xrightarrow{} 0}{\underset{\to 1}{\cos x} + \underset{\to 1}{\cos x} - \underset{\to 0}{x\sin x}} = \frac{0}{2} = 0$$

Indeterminate Powers: 0^0, 1^∞, and ∞^0 ⇒ **Use a logarithmic "trick".**

These indeterminate powers arise if the function is of the form $[f(x)]^{g(x)}$, where $f(x) > 0$.

We will apply the natural logarithm function.

First recall a couple of logarithmic rules: $e^{\ln x} = x$ and $\ln x^p = \underline{\underline{p}} \ln x$.

The 1^∞ Case

Example 5: Evaluate the limit, if it exists. $\lim\limits_{x\to1} x^{\frac{3}{x-1}}$ $= 1^\infty$

$$\lim_{x\to1} e^{\ln\left(x^{\frac{3}{x-1}}\right)} = e^{\lim\limits_{x\to1}\ln\left(x^{\frac{3}{x-1}}\right)} = \boxed{e^3}$$

$$\lim_{x\to1}\ln\left(x^{\frac{3}{x-1}}\right) = \lim_{x\to1}\left(\overset{\nearrow\infty}{\frac{3}{x-1}}\cdot\overset{\nearrow 0}{\ln x}\right)$$

$$= \lim_{x\to1}\left(\frac{\ln x \nearrow 0}{\frac{x-1}{3} \to 0}\right) = \lim_{x\to1}\frac{\frac{1}{x}}{\frac{1}{3}} = \frac{1}{\frac{1}{3}} = 3$$

The 0^0 Case

Example 6: Evaluate the limit if it exists. $\lim_{x\to 0^+}(\sin x)^x$ $= 0^0$

$$\lim_{x\to 0^+} e^{\ln(\sin x)^x} = e^{\lim_{x\to 0^+}\ln(\sin x)^x} = e^0 = \boxed{1}$$

$$\lim_{x\to 0^+} \ln(\sin x)^{x} = \lim_{x\to 0^+} \overset{\nearrow 0}{x}\cdot\overset{\nearrow -\infty}{\ln(\sin x)}$$

$$= \lim_{x\to 0^+} \frac{\ln(\sin x) \nearrow -\infty}{\frac{1}{x} \searrow \infty} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} -\frac{x^2\cos x \nearrow 0}{\sin x \searrow 0}$$

$$= \lim_{x\to 0^+} -\frac{2x\cos x - x^2\sin x}{\cos x} = \frac{0}{1} = 0$$

The ∞^0 Case

Example 7: Evaluate the limit if it exists. $\lim\limits_{x\to\infty}(x^4+1)^{\frac{1}{\ln x}} = \infty^0$

$$\lim_{x\to\infty} e^{\ln(x^4+1)^{\frac{1}{\ln x}}} = e^{\lim\limits_{x\to\infty} \ln(x^4+1)^{1/\ln x}} = \boxed{e^4}$$

$$\lim_{x\to\infty} \ln(x^4+1)^{\frac{1}{\ln x}} = \lim_{x\to\infty} \overset{\nearrow 0}{\frac{1}{\ln x}} \cdot \overset{\nearrow \infty}{\ln(x^4+1)}$$

$$= \lim_{x\to\infty} \frac{\ln(x^4+1) \nearrow \infty}{\ln x \searrow \infty} = \lim_{x\to\infty} \frac{\frac{4x^3}{x^4+1}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{4x^3}{x^4+1}\cdot x = \lim_{x\to\infty} \frac{4x^4}{x^4+1} = 4$$

Summary

- **Indeterminate Forms $0/0$ or $\infty/\infty \Rightarrow$ Use L'Hospital's rule**
- **Indeterminate Forms $0 \cdot \infty$ or $\infty - \infty \Rightarrow$ Rewrite the function so that you get $0/0$ or ∞/∞ and then use L'Hospital's rule.**
- **Indeterminate Powers: $0^0, 1^\infty$, and $\infty^0 \Rightarrow$ Use a logarithmic "trick" to see which indeterminate form you have and only use L'Hospital's rule when you get $0/0$ or ∞/∞.**
- **The following cases are "determinate" (do not use L'Hospital's rule):**
  $\infty + \infty \to \infty$
  $-\infty - \infty \to -\infty$
  $0^\infty \to 0$
  $0^{-\infty} \to \infty$