Section 6.3 – Basic Integration Rules

The notation $\int f(x)dx$ is used for an antiderivative of f and called an **indefinite integral**.

$$\int f(x)\,dx = F(x) \text{ means } F'(x) = f(x)$$

In general, to find $\int f(x)dx$, we find an antiderivative of $f(x)$, say $F(x)$, and then we write the indefinite integral as $\int f(x)dx = F(x) + C$. Here, C is called the **constant of integration**.

If given $\int_a^b f(x)dx$, this is a **definite integral** and to evaluate we'll use Part 2 of the

Fundamental Theorem of Calculus: $\int_a^b f(x)dx = F(b) - F(a)$

The Constant Rule for Integrals

$\int k dx = k \cdot x + C$, where k is a constant number.

Example 1: Find of each of the following integrals.

a. $\int 10dx$ $= 10x + C$

b. $\int_1^4 \pi dx$ $= \pi x \Big|_1^4 = 4\pi - \pi = 3\pi$

$= (\pi x + C)\Big|_1^4 = (4\pi + C) - (\pi + C)$

$= 4\pi + C - \pi - C$

$= 3\pi$

The Power Rule for Integrals

$\int x^r dx = \frac{x^{r+1}}{r+1} + C$, where $r \neq -1$.

Example 2: Find of each of the following integrals.

a. $\int x^4 dx$

$= \frac{x^5}{5} + C$

b. $\int_0^1 \frac{1}{\sqrt{x}} dx$ $= \int_0^1 x^{-1/2} dx = \frac{x^{1/2}}{1/2}\Big|_0^1 = 2\sqrt{x}\Big|_0^1$

$= 2\sqrt{1} - 2\sqrt{0} = 2$

The Constant Multiple of a Function for Integrals

$\int k \cdot f(x) dx = k \int f(x) dx$, where k is a constant number.

Example 3: $\int_{-1}^{1} 4x^7 dx$ $= \frac{\cancel{4} x^8}{\cancel{8}\,2}\Big|_{-1}^{1} = \frac{x^8}{2}\Big|_{-1}^{1} = \frac{(1)^8}{2} - \frac{(-1)^8}{2} = 0$

If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x) dx = 0$

$f(-x) = -f(x)$

$f(x) = 4x^7$

$f(-x) = 4(-x)^7 = -4x^7$

The Sum/Difference of Functions for Integrals

$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Example 4: $\int_0^1 (5x^4 + 4x + 7)dx$ $= \left(\frac{\cancel{5}x^5}{\cancel{5}} + \frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}} + 7x\right)\Big|_0^1$

$= (x^5 + 2x^2 + 7x)\Big|_0^1$

$= 1^5 + 2\cdot 1^2 + 7\cdot 1 - 0$

$= 1 + 2 + 7 = 10$

Example 5: $\int \frac{x+2\sqrt{x}}{\sqrt{x}}dx$ $= \int \left(\frac{x}{\sqrt{x}} + 2\right)dx = \int (x^{1/2} + 2)\,dx$

$= \frac{x^{3/2}}{3/2} + 2x + C$

$= \frac{2}{3}x^{3/2} + 2x + C$

$x^n \cdot x^m = x^{n+m}$

Example 6:

a. $\int \sqrt[3]{x}(x-4)dx$ $= \int x^{1/3}(x-4)\,dx = \int \left(x^{4/3} - 4x^{1/3}\right)dx$

$= \dfrac{x^{7/3}}{7/3} - 4\dfrac{x^{4/3}}{4/3} + C = \dfrac{3}{7}x^{7/3} - \cancel{4}\cdot\dfrac{3}{\cancel{4}}x^{4/3} + C$

$= \dfrac{3}{7}x^{7/3} - 3x^{4/3} + C$

b. $\int 2x(x-1)(x+1)dx$ expand! $2x(x^2-1) = 2x^3 - 2x$

$= \int (2x^3 - 2x)\,dx = \dfrac{\cancel{2}x^4}{\cancel{4}\,2} - \dfrac{\cancel{2}x^2}{\cancel{2}} + C$

$= \dfrac{x^4}{2} - x^2 + C$

c. $\int \frac{5x^3+x}{x^3}dx$ divide!

$= \int (5 + x^{-2})\,dx = 5x + \dfrac{x^{-1}}{-1} + C$

$= 5x - \dfrac{1}{x} + C$

Other times we are given the derivative and an initial value and we are asked to find the **original function.**

Example 7: Given $f'(x) = 2x + 2$, $f(1) = 5$, find $f(x)$.

$$f(x) = \int (2x+2)\,dx = \frac{2x^2}{2} + 2x + C = x^2 + 2x + C$$

$$1^2 + 2(1) + C = 5$$

$$1 + 2 + C = 5$$

$$C = 2$$

$$\boxed{f(x) = x^2 + 2x + 2}$$

Example 8: Given $f''(x) = 6x + 2$, $f'(0) = 2$, $f(0) = 10$, find $f(x)$.

$$f'(x) = \int (6x+2)\,dx = \frac{6x^2}{2} + 2x + C = 3x^2 + 2x + C$$

$$f'(0) = 3 \cdot 0^2 + 2 \cdot 0 + C = 2 \qquad C = 2$$

$$f'(x) = 3x^2 + 2x + 2$$

$$f(x) = \int (3x^2 + 2x + 2)\,dx = \frac{3 \cdot x^3}{3} + \frac{2x^2}{2} + 2x + D$$

$$f(x) = x^3 + x^2 + 2x + D$$

$$f(0) = 0^3 + 0^2 + 2 \cdot 0 + D = 10 \qquad D = 10$$

$$\boxed{f(x) = x^3 + x^2 + 2x + 10}$$

Integrals of Basic Trigonometric Functions:

$\int \sin x \, dx = -\cos x + C$

$\int \cos x \, dx = \sin x + C$

$\int \sec^2 x \, dx = \tan x + C$

$\int \csc^2 x \, dx = -\cot x + C$

$\int \sec x \tan x \, dx = \sec x + C$

$\int \csc x \cot x \, dx = -\csc x + C$

Example 9: $\int \sin x (\csc x + \cot x) \, dx$ $= \int \left[\cancel{\sin x} \cdot \frac{1}{\cancel{\sin x}} + \cancel{\sin x} \cdot \frac{\cos x}{\cancel{\sin x}} \right] dx$

$= \int (1 + \cos x) dx = x + \sin x + C$

Example 10: $\int_0^{\pi/3} \sec x \tan x \, dx$ $= \sec x \Big|_0^{\pi/3}$

$= \sec \pi/3 - \sec 0$

$= \frac{1}{\cos(\pi/3)} - \frac{1}{\cos(0)} = 2 - 1 = 1$

Example 11: Given $f'(x) = -3 \sin x$, $f(\pi) = -1$, find $f(x)$.

$\int (-3 \sin x) dx = -3(-\cos x) + C = 3\cos x + C$

$f(\pi) = 3\cos \pi + C = -1$

$-3 + C = -1$

$C = 2$

$\boxed{f(x) = 3\cos x + 2}$

Integral of $\frac{1}{x}$

$$\int \frac{1}{x} dx = \ln|x| + C$$

Example 12: $\int_1^2 \frac{x^2-2}{x} dx$ $= \int_1^2 \left(x - 2 \cdot \frac{1}{x}\right) dx$

$$= \left(\frac{x^2}{2} - 2\ln|x|\right)\Big|_1^2$$

$$= \left(\frac{2^2}{2} - 2\ln 2\right) - \left(\frac{1^2}{2} - 2\ln|1|\right)$$

$$= 2 - \ln 4 - \frac{1}{2} + 2(0)$$

$$= \frac{3}{2} - \ln 4$$

Integrals of Exponential Functions

$$\int e^x dx = e^x + C$$

$\int a^x dx = \frac{a^x}{\ln a} + C$, where $a > 0,\ a \neq 1$

Example 13: $\int (2^x + 5e^x) dx$

$$= \frac{2^x}{\ln 2} + 5e^x + C$$

Integrals of the Hyperbolic Functions

$\int \sinh x\, dx = \cosh x + C$

$\int \cosh x\, dx = \sinh x + C$

Recall: $\cosh x = \frac{e^x + e^{-x}}{2}$; $\sinh x = \frac{e^x - e^{-x}}{2}$

Example 14: $\int_0^1 2 \sinh x\, dx$ $= 2\cosh x \Big|_0^1 = 2\cosh 1 - 2\cosh 0$

$= 2\left(\frac{e + \frac{1}{e}}{2}\right) - 2(1) = e + \frac{1}{e} - 2$

$\cosh 1 = \frac{e^1 + e^{-1}}{2} = \frac{e + \frac{1}{e}}{2}$

$\cosh 0 = \frac{e^0 + e^0}{2} = \frac{1+1}{2} = 1$

Integrals Resulting in Inverse Trigonometric Functions Memorize!

$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

$\int \frac{1}{1+x^2} dx = \arctan x + C$

$\int \frac{1}{|x|\sqrt{x^2-1}} dx = \operatorname{arcsec} x + C$

Example 15: $\int_{1/2}^{\sqrt{2}/2} \frac{2}{\sqrt{1-x^2}} dx$ $= 2\arcsin x \Big|_{1/2}^{\sqrt{2}/2}$

$= 2\arcsin\left(\frac{\sqrt{2}}{2}\right) - 2\arcsin\left(\frac{1}{2}\right)$

$= \cancel{2} \cdot \frac{\pi}{\cancel{4}\,2} - \cancel{2} \cdot \frac{\pi}{\cancel{6}\,3} = \frac{3 \cdot \pi}{3 \cdot 2} - \frac{\pi \cdot 2}{3 \cdot 2} = \frac{3\pi - 2\pi}{6}$

$= \frac{\pi}{6}$

Integrating Piece-wise Defined Functions

Example 16: Let $f(x) = \begin{cases} x+2, & -2 \le x \le 0 \\ 2, & 0 < x \le 1 \\ 4-2x, & 1 < x \le 2 \end{cases}$

Note how the function changes over the specified domain!

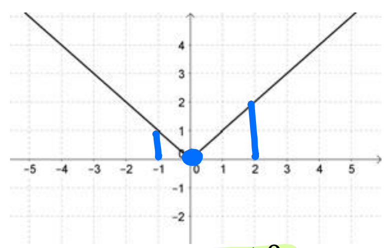
Set-up the integral needed to integrate $\int_{-2}^{2} f(x)dx$.

$$= \int_{-2}^{0} (x+2)\,dx + \int_{0}^{1} 2\,dx + \int_{1}^{2} (4-2x)\,dx$$

Integrals Involving Absolute Value

Example 17: $\int_{-1}^{2} |x| dx$

Recall that $y = |x|$ is a piecewise function!



$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \ge 0 \end{cases}$

$$= \int_{-1}^{0} -x\,dx + \int_{0}^{2} x\,dx$$

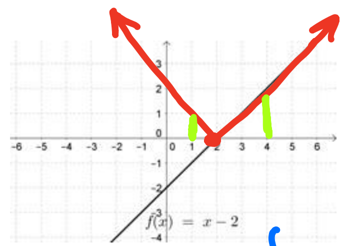
$$= -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{2}$$

$$= \left(0 - \left(-\frac{(-1)^2}{2}\right)\right) + \left(\frac{2^2}{2} - 0\right)$$

$$= \frac{1}{2} + 2 = 2\frac{1}{2} = \frac{5}{2}$$

Example 18:

a. $\int_1^4 |x-2|dx$



$$= \int_1^2 -(x-2)dx + \int_2^4 (x-2)\,dx$$

$$|x-2| = \begin{cases} -(x-2), & x < 2 \\ x-2, & x \geq 2 \end{cases}$$

$$= \int_1^2 (-x+2)dx + \int_2^4 (x-2)dx$$

$$= \left(-\frac{x^2}{2} + 2x\right)\Big|_1^2 + \left(\frac{x^2}{2} - 2x\right)\Big|_2^4$$

$$= \left[-\frac{2^2}{2} + 2(2) + \frac{1^2}{2} - 2(1)\right] + \left[\frac{4^2}{2} - 2(4) - \frac{2^2}{2} + 2(2)\right]$$

$$= 2.5$$

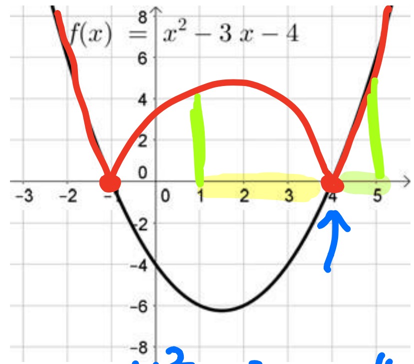
b. $\int_5^6 |x-2|dx$ $= \int_5^6 (x-2)\,dx$

$$= \left(\frac{x^2}{2} - 2x\right)\Big|_5^6 = \left[\frac{6^2}{2} - 2(6)\right] - \left[\frac{5^2}{2} - 2(5)\right]$$

$$= 3.5$$

Example 19: Set up the following integrals.

a. $\int_1^5 |x^2 - 3x - 4| \, dx$



$x^2 - 3x - 4 = 0$

$(x+1)(x-4) = 0$

$x = -1 \quad x = 4$

$$= \int_1^4 -(x^2 - 3x - 4)\, dx$$

$$+ \int_4^5 (x^2 - 3x - 4)\, dx$$

$$= \frac{49}{3}$$

b. $\int_1^5 |x^2 + 4| \, dx$ $= \int_1^5 (x^2 + 4)\, dx$