Section 2.1
Counting Techniques

Combinatorics is the study of the number of ways a set of objects can be arranged, combined, or chosen; or the number of ways a succession of events can occur. Each result is called an **outcome**. The collection of all possible outcomes is the **sample space**. An **event** is a subset of outcomes. When several events occur together, we have a **compound event**.

The **Fundamental Counting Principle** states that the total number of a ways a compound event may occur is $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_i$ where n_1 represents the number of ways the first event may occur, n_2 represents the number of ways the second event may occur, and so on.

Example 1: The Burger Bar offers the following items on its menu:

2	4	3	4
Burger	Sides	Beverages	Desserts
Single Meat	Fries	Tea	Cheesecake
Double Meat	Onion Rings	Coffee	Brownie
	Fruit Bowl	Soda	Cookie
	Cheddar Peppers		Ice Cream Cone

If a customer chooses 1 item from each category, how many meals can be made? List 1 meal possible.

$2(4)(3)(4) = 96$

SM
Fruit Bowl
Coffee
Cookie
] 1 of 96

Example 2: A license plate consists of 3 letters followed by 4 digits. How many license plates are possible if the first letter cannot be O, repetition of letters is allowed, but digits may not repeat?

no "O"

$\underset{L}{\underline{25}} * \underset{L}{\underline{26}} * \underset{L}{\underline{26}} * \underset{D}{\underline{10}} * \underset{D}{\underline{9}} * \underset{D}{\underline{8}} * \underset{D}{\underline{7}} = \boxed{85{,}176{,}000}$

Example 3: How many ways can the letters of the word VOWEL be arranged if the first letter cannot be a vowel?

1 2 3 4 5

$\underline{3} * \underline{4} * \underline{3} * \underline{2} \times \underline{1} = \boxed{72}$

Permutations

n-Factorial: For any natural number n, $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.
$0! = 1$
R command: factorial( )

$5! = 5*4*3*2*1 = 120$

A **permutation** of a set is arranging the elements of the set with regard to order.

Example: My previous pin number was 2468, now it's 8642.

Formula: ${}_nP_r = \boxed{P(n,r) = \frac{n!}{(n-r)!}}$, $r \le n$, where n is the number of distinct objects and r is the number of distinct objects taken r at a time.

Example 4: Seven people arrive at a ticket counter at the same time to buy concert tickets. In how many ways can they line up to purchase their tickets?

3-5-08

$\underline{7} * \underline{6} * \underline{5} * \underline{4} * \underline{3} * \underline{2} * \underline{1} = 7!$

Command: Answer:

factorial(7) $\longrightarrow$ 5040

factorial(7)/factorial(0)

$P(7,7) = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7!}{1} = 7!$

Example 5: In how many ways can 3 of the six symbols, @, &, %, $, *, # be arranged on an ID tag?

$n = 6$
$r = 3$

$P(6,3) = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 120$

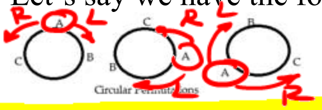
Command: Answer:

factorial(6)/factorial(3) 120

$\underline{6} * \underline{5} * \underline{4}$

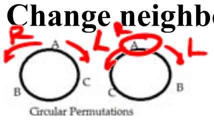
Circular Permutations

Let's say we have the following situation…



How can persons *A, B, C* be arranged around a circle?

Not in the three ways as shown above because each one of *A, B, C* has the same neighbor! Without changing neighbor**, only changing seats will not change the circular permutation**. **Change neighbors and you will change the circular permutation.** As follows:



So, three persons *A, B, C* can only be arranged in 2 ways around a circle. Hence, *n* different things can be arranged **around a circle in (*n* – 1)! ways**. Whereas, *n* different things can be arranged **in a line n! ways**.

http://www.math-for-all-grades.com/CircularPermutation.html

Example 6: In how many ways can 12 people (n) be seated around a circular table?

Command: factorial(11)

Answer: 39,916,800

Formula: Permutations of n objects, not all distinct

Given a set of n objects in which n_1 objects are alike and of one kind, n_2 objects are alike and of another kind,…, and, finally, n_r objects are alike and of yet another kind so that

$$n_1 + n_2 + ... + n_r = n$$

then the number of permutations of these *n* objects taken *n* at a time is given by

$$\boxed{\frac{n!}{n_1!n_2!\cdots n_r!}}$$ ← multiplied!

Example 7: How many arrangements can be made using all of the letters in the word MISSISSIPPI?

n = total number of objects = 11

letter *occurs*

M => $n_1 = 1$

I => $n_2 = 4$

S => $n_3 = 4$

P => $n_4 = 2$

$$\frac{11!}{\cancel{1}!\,4!\,4!\,2!} \qquad 1! = 1$$

Command: factorial(11)/(factorial(4)*factorial(4)*factorial(2))

Answer: 34,650

Combinations

A **combination** of a set is arranging the elements of the set without regard to order.

Example: The marinade for my steak contains soy sauce, worchester sauce and a secret seasoning.

Formula: ${}_nC_r = \boxed{C(n,r) = \frac{n!}{r!(n-r)!}}$, $r \le n$, where n is the number of distinct objects and r is the number of distinct objects taken r at a time. R Command: choose(n, r)

Example 8: An organization needs to make up a social committee. If the organization has 25 members, in how many ways can a 10 person committee be made?

C(25, 10)

Command: choose(25,10)

Answer: 3,268,760

Example 9: A committee of 16 people, 7 women and 9 men, is forming a 7- member subcommittee that must consist of 3 women and 4 men. In how many ways can the subcommittee be formed?

16

7W → 3W

9M → 4M

C(7,3) * C(9,4)

Command: choose(7,3) * choose(9,4)

Answer: 4410

Example 10: A tumbler contains 30 ~~cubes~~ balls, in which 10 are blue, 10 are yellow, and 10 are red. The balls of each color are lettered A – J. You choose 6 balls at random from the tumbler. How many selections consist of exactly 3 balls with the same letter?

A-J → 10 letters

$10 * C(27,3)$

30

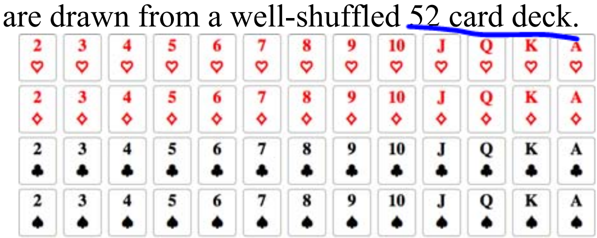
3 same letter → 3

27 different → 3

Command: 10 * choose (27,3)

Answer: 29,250

Try this one: Five cards are drawn from a well-shuffled 52 card deck.



a. In how many ways can the five cards be drawn?

$C(52,5)$

Command: choose (52,5)

Answer: 2,598,960

b. In how many ways can four Queens be drawn?

$C(4,4) * C(48,1)$

52

4Q → 4Q

48 other cards → 1 Card

Command: choose (4,4) * choose (48,1)

Answer: 48

add

c. In how many ways can four Queens or four Kings be drawn?

C(4,4) C(48,1) + C(4,4) C(48,1)

Command:

choose(4,4) * choose(48,1)
+ choose(4,4) * choose(48,1)

Answer:

96

d. In how many ways can any four of a kind be drawn?

There are 13 kinds in a deck

$13 * \underbrace{C(4,4)\,C(48,1)}_{48}$

13 * 48

Command:

Answer:

624