Discrete Mathematics
Propositional Logic

What is a proposition?

Definition: A proposition (or a statement) is a sentence that is either true or false, but not both.

Examples of Propositions:

a. Austin is the capital of Texas. T
b. Texas is the largest state of the United States. F
c. $1 + 0 = 1$ T

Examples that are NOT Propositions:

a. Watch out!
b. What time is it?
c. $x + 3 = 5$ $x = 2$ T, $x = 0$ F

- Letters are used to denote propositions: $p, q, r, s \ldots$
- The truth value of a proposition that is always true denoted by T, the truth value of a proposition that is always false denoted by F.

New propositions (compound propositions) can be formed from existing propositions using logical operators.

Definition: Let p be a proposition. The *negation of p*, denoted by $\neg p$ ($\sim$), the statement

"It is not the case that p."

Examples:

a. Proposition: A triangle has three sides. T
   Negation: It is not the case that triangle has three sides. F
   Negation in simple English: A triangle does not have three sides.

b. Proposition: All fish can swim. T
   Negation: It is not the case that all fish can swim.
   Negation in simple English: Some fish cannot swim. F

   All $\leftrightarrow$ Some

c. Proposition: $2 + 3 > 5$ F
   Negation: $2+3 \leq 5$ T

   $=$ $<$ $>$

A proposition and its negation have OPPOSITE truth values!

Construct a truth table for the negation of p.

$\boldsymbol{p}$	$\neg \boldsymbol{p}$
T	F
F	T

Definition: Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction is true when BOTH p and q are true and is false otherwise.

Example: Construct a truth table for the conjunction.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$
T	T	T
T	F	F
F	T	F
F	F	F

Example: Find the conjunction of the following propositions and determine its truth value.

a. p: All birds can fly.
   q: $2 + 3 = 5$

   Conjunction:

F

All birds can fly <u>and</u> $2+3=5$.

T

False

Definition: Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction is false when BOTH p and q are false and is true otherwise.

Example: Construct the truth table for the disjunction.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \vee q}$
T	T	T
T	F	T
F	T	T
F	F	F

Example: Find the disjunction of the following propositions and determine its truth value.

a. p: Triangles are square.
   q: Circles are round.
   Disjunction:

Triangles are square or circles are round.

F T

True

Definition: Let p and q be propositions. The *exclusive* of p and q, denoted by $p \oplus q$, is the proposition "p or q, but not both." The exclusive is true when ONE of p and q is true and is false otherwise.

Example:

Students who have taken calculus or computer science can take this class. Disj

Soup or salad comes with this entrée. Exc.

Example: Construct the truth table for the exclusive.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Definition: Let p and q be propositions. The *conditional statement (implication)* $p \rightarrow q$ is the proposition "if p, then q."

The conditional statement $p \rightarrow q$ is false then p is true and q is false, and true otherwise.

In the conditional statement $p \rightarrow q$, p is called hypothesis and q is called conclusion.

Example: The Truth Table for the Conditional Statement $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- Connection between the hypothesis and conclusion is NOT necessary.
- Think: Implication = Contract.

Example:

T F

a. If you get 100% on the final, then you will get an A. False

F T

b. If the Moon made of cheese, then 1 + 1 = 2. True

p q

Different Ways of Expressing $p \to q$

if p**, then** q

if p, q

q **unless** $\neg p$

q **if** p

q **whenever** p

q **follows from** p

p **implies** q

p **only if** q

q **when** p

p **is sufficient for** q

q **is necessary for** p

a necessary condition for p **is** q

a sufficient condition for q **is** p

Example:
Write the statement in the "If…, then…" form.

a. It is hot whenever it is sunny.

q p

If it is sunny, then it is hot.

b. To get a good grade it is necessary that you study.

Studing is necessary for good grades.

q p

If you get good grades, then you study.

$p \to q$

Definitions:
The proposition $q \to p$ is called *converse*.
The proposition $\neg p \to \neg q$ is called *inverse*.
The proposition $\neg q \to \neg p$ is called *contrapositive*.

Example: Write the converse, inverse, and contrapositive for the following statement.

a. If $\underbrace{\overset{F}{3 \geq 5}}_{p}$, then $\underbrace{\overset{F}{7 > 7}}_{q}$. True

Converse:

If $\underset{F}{7 > 7}$, then $\underset{F}{3 \geq 5}$. True

Inverse:

If $\underset{T}{3 < 5}$, then $\underset{T}{7 \leq 7}$. True

Contrapositive:

If $\underset{T}{7 \leq 7}$, then $\underset{T}{3 < 5}$. True

b. I come to class whenever there is going to be a quiz.
(Hint: Rewrite the proposition in the "if, then" form)

q p

If there is going to be a quiz, then I come to class.

Converse:

If I come to class, then there is going to be a quiz.

Inverse:

If there is not going to be a quiz, then I do not come to class.

Contrapositive:

If I do not come to class, then there is not going to be a quiz.

Definition: Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the SAME truth values, and is false otherwise.

"if and only if" = "iff"

Example: You can drive a car if and only if your gas tank is not empty.

Example: The Truth Table for the Biconditional Statement $\leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Expressing the Biconditional
p **is necessary and sufficient for** q
if p **then** q **, and conversely**
p **iff** q

Truth Tables for Compound Propositions
Construction of a truth table:

1. Rows
    - Need a row for every possible combination of values for the atomic propositions. 2^n
2. Columns
    - Need a column for the compound proposition (usually at far right)
    - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.

Example: Construct a truth table for $(p \vee \neg q) \rightarrow (p \wedge q)$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Equivalent Propositions

Definition: Two propositions are equivalent if they always have the same truth value.

$p \rightarrow q$ $\qquad$ $\neg q \rightarrow \neg p$

Example: Show using a truth table that the conditional is equivalent to the contrapositive.

p	q	$p \rightarrow q$	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

same

Precedence of Logical Operators

Operator	Precedence
¬	**1**
∧	**2**
∨	**3**
→	**4**
↔	**5**

Example:

2 3 1

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$

If the intended meaning is $p \vee (q \rightarrow \neg r)$ then parentheses must be used.