Discrete Mathematics
Predicates and Quantifiers

Predicates

Propositional logic is not enough to express the meaning of all statements in mathematics and natural language.

Examples:
Is “$x > 1$” True or False?

Is “$x$ is a great tennis player” True or False?

Predicate Logic
- Variables: $x, y, z$, etc.
- Predicates: $P(x), Q(x)$, etc.
- Quantifiers: Universal and Existential.
- Connectives from propositional logic carry over to predicate logic.

A predicate $P(x)$ is a declarative sentence whose truth value depends on one or more variables.

$P(x)$ is also said to be the value of the propositional function $P$ at $x$.

$P(x)$ becomes a proposition when a value of $x$ is assigned from the domain $U$.

Examples (Propositional Functions):

1. Let $P(x)$ be “$x \geq 1$.” Determine the truth value of
   a. $P(2)$
   b. $P(-2) \rightarrow P(1)$

2. Let $R(x, y, z)$ be “$x + y = z$.” Find these truth values:
   a. $R(2, -1, 5)$
   b. $R(x, 3, z)$
Quantifiers

We need quantifiers to express the meaning of English words including all and some:

- “All students in this class are computer science majors”
- “There is a math major student in this class”

The two most important quantifiers are:

- Universal Quantifier, “For all,” symbol: ∀
- Existential Quantifier, “There exists,” symbol: ∃

We write as in ∀x P(x) and ∃x P(x).

- ∀x P(x) asserts P(x) is true for every x in the domain.
  
  If = {x₁, x₂, ..., xₙ }, then   ∀x P(x) = P(x₁) ∧ P(x₂) ... ∧ P(xₙ).

- ∃x P(x) asserts P(x) is true for some x in the domain.
  
  If = {x₁, x₂, ..., xₙ }, then   ∃x P(x) = P(x₁) ∨ P(x₂) ... ∨ P(xₙ).

Examples:
1. Let P(x): “x > −x” with the domain of all positive real numbers. Find the truth value of ∀x P(x).

2. Let P(x): “x > −x” with the domain of all real numbers. Find the truth value of ∀x P(x).

- The truth value of ∃x P(x) and ∀x P(x) depends BOTH on the propositional function P(x) and on the domain U.

<table>
<thead>
<tr>
<th>Quantifiers</th>
<th>When True?</th>
<th>When False?</th>
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<tr>
<td>∀x P(x)</td>
<td>P(x) is true for every x.</td>
<td>There is an x for which P(x) is false.</td>
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<tr>
<td>∃x P(x)</td>
<td>There is an x for which P(x) is true.</td>
<td>P(x) is false for every x.</td>
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Example: Suppose the domain of the propositional function \( P(x): x^2 \leq x \) consists of \( \{1, 2, 3\} \). Write out each of the following propositions using conjunction or disjunction and determine its truth value.

1. \( \forall x \, P(x) \)

2. \( \exists x \, P(x) \)

An element for which \( P(x) \) is false is called a **counterexample of** \( \forall x \, P(x) \)

**Precedence of Quantifiers**

The quantifiers \( \forall \) and \( \exists \) have higher precedence than all the logical operators.

**Example:** \( \forall x \, P(x) \lor Q(x) \) means \( (\forall x \, P(x)) \lor Q(x) \). \( \forall x \, (P(x) \lor Q(x)) \) means something different.

**Negating Quantifiers**

**De Morgan laws for quantifiers** (the rules for negating quantifiers) are:

\[ \neg \forall x \, P(x) \equiv \exists x \, \neg P(x) \]

\[ \neg \exists x \, P(x) \equiv \forall x \, \neg P(x) \]

**Example:** Express each of these statements using quantifiers. Then form a negation of the statement, so that no negation is left of a quantifier. Next, express the negation in simple English.

1. “Some old dogs can learn new tricks.”
2. “Every bird can fly.”

3. $\forall x (x^2 > x)$
Translating from English into Logical Expressions

**Examples:** Translate the statements into the logical symbols. Let $x$ be in set of all students in this class.

1. Someone in your class can speak Hindi.
2. Everyone in your class is friendly.
3. There is a student in your class who was not born in California.

$$H(x) = "x speaks Hindi", \quad F(x) = "x is friendly," \quad C(x) = "x was born in California."$$

**Example:** Translate the following sentence into predicate logic and give its negation:

“Every student in this class has taken a course in Java.”

**Solution:**

First, decide on the domain $U$!

**Solution 1:** If $U$ is all students in this class, define a propositional function $J(x)$ denoting “$x$ has taken a course in Java” and translate as

**Solution 2:** But if $U$ is all people, also define a propositional function $S(x)$ denoting “$x$ is a student in this class” and translate as
**Example:** Translate the following sentence into predicate logic:

“Some student in this class has taken a course in Java.”

**Solution:**
First, decide on the domain $U$!

**Solution 1:** If $U$ is all students in this class, translate as

**Solution 2:** But if $U$ is all people, then translate as