Discrete Mathematics
Nested Quantifiers

Nested quantifiers

Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

Example: "Every real number has an additive inverse" is translated as $\forall x\, \exists y(x + y = 0)$, where the domains of x and y are the real numbers.

Example: Translate into English the statement

$$\forall x \forall y((x < 0) \wedge (y < 0) \rightarrow (xy > 0))$$

where the domains of x and y are the real numbers.

Order of quantifiers

Examples:

1. Let $P(x, y)$ be the predicate "$x + y = y + x$" Assume that U is the real numbers. Determine the truth value of $\forall x \forall y\, P(x, y)$ and $\forall y \forall x\, P(x, y)$.

2. Let $Q(x, y)$ be the predicate "$x + y = 0$" Assume that U is the real numbers. Determine the truth value of $\forall x \exists y Q(x, y)$ and $\exists y \forall x Q(x, y)$.

Pay attention to the order of quantifiers when there are different ones in one statement!

Example: Let U be the real numbers. Define $P(x, y) : x \cdot y = 0$.

What is the truth value of the following?

1. $\forall x \forall y P(x, y)$
2. $\forall x \exists y P(x, y)$
3. $\exists x \forall y\, P(x, y)$
4. $\exists x \exists y P(x, y)$

Example: Let U be the real numbers. Define $P(x, y) : x / y = 1$

What is the truth value of the following?

1. $\forall x \forall y P(x, y)$
2. $\forall x \exists y P(x, y)$
3. $\exists x \forall y\ P(x, y)$
4. $\exists x \exists y P(x, y)$

Translating Mathematical Statements into Statements with Nested Quantifiers

Example: Express the following statement using mathematical and logical operators, predicates, and quantifiers. The domain consists of all integers.

1. The sum of two negative integers is negative.

2. The difference of two positive integers is not necessarily positive.

Translating Nested Quantifiers into English

Example: Translate the statement $\forall x\ (C(x) \vee \exists y\ (C(y) \wedge F(x, y)))$

where $C(x)$ is "x has a computer," and $F(x, y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

Example: Translate the statement $\exists x \forall y \, \forall z \, ((F(x,y) \wedge \, F(x,z) \, \wedge \, (y \, \neq z)) \rightarrow \neg F(y,z))$

Examples: Express in predicate logic using the following predicates.

$$B(x,y) = x \; is \; a \; brother \; of \; y, \qquad S(x,y) = x \; is \; a \; sibling \; of \; y$$

$$L(x,y) = x \; loves \; y$$

1. "Brothers are siblings."

2. "Everybody loves somebody."

3. "There is someone who is loved by everyone."

4. "There is someone who loves someone."

5. "Everyone loves himself"

Negating Nested Quantifiers

- Use De Morgan's Laws to move the negation as far inwards as possible.

Example: Express the negation of the statement $\forall x \exists y (xy = 1)$ so that no negation precedes a quantifier.