Discrete Mathematics
Nested Quantifiers

Nested quantifiers
Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

Example: "Every real number has an additive inverse" is translated as $\forall x\, \exists y(x + y = 0)$, where the domains of x and y are the real numbers.

Example: Translate into English the statement

$$\forall x \forall y((x < 0) \land (y < 0) \rightarrow (xy > 0))$$

where the domains of x and y are the real numbers.

Negative times negative is positive.

Order of quantifiers

Examples:

1. Let $P(x, y)$ be the predicate "$x + y = y + x$" Assume that U is the real numbers. Determine the truth value of $\forall x \forall y\, P(x, y)$ and $\forall y \forall x\, P(x, y)$.
   True True

2. Let $Q(x, y)$ be the predicate "$x + y = 0$" Assume that U is the real numbers. Determine the truth value of $\forall x \exists y Q(x, y)$ and $\exists y \forall x Q(x, y)$.
   True False!

Pay attention to the order of quantifiers when there are different ones in one statement!

Example: Let U be the real numbers. Define $P(x, y) : x \cdot y = 0$.

What is the truth value of the following?

1. $\forall x \forall y P(x, y)$ False
   $2 \cdot 3 \neq 0$
2. $\forall x \exists y P(x, y)$ True
   $y = 0$
3. $\exists x \forall y\, P(x, y)$ True
   $x = 0$
4. $\exists x \exists y P(x, y)$ True

Example: Let U be the real numbers. Define $P(x, y)$: $x / y = 1$

What is the truth value of the following?

1. $\forall x \forall y P(x, y)$ False $\begin{pmatrix} x = 6 \\ y = 2 \end{pmatrix}$
2. $\forall x \exists y P(x, y)$ False $(x = 0)$
3. $\exists x \forall y\, P(x, y)$ False $(y = 0)$
4. $\exists x \exists y P(x, y)$ True (Ex: 1 & 1, 2 & 2, -3 & -3...)

Translating Mathematical Statements into Statements with Nested Quantifiers

Example: Express the following statement using mathematical and logical operators, predicates, and quantifiers. The domain consists of all integers.

1. The sum of two negative integers is negative.

$$\forall x \forall y\, ((x < 0) \wedge (y < 0) \rightarrow (x + y) < 0)$$

2. The difference of two positive integers is not necessarily positive.

$$\neg \forall x \forall y\, ((x > 0) \wedge (y > 0) \rightarrow (x + y) > 0)$$

Translating Nested Quantifiers into English

Example: Translate the statement $\forall x\ (C(x) \vee \exists y\, (C(y) \wedge F(x, y)))$ (or ... and)

where $C(x)$ is "x has a computer," and $F(x, y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

Every student in your school has a computer or has a friend (from his/her school) that has a computer.

Example: Translate the statement $\exists x \forall y\, \forall z\, ((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z))$

There is a student in your school such that none of his/her friends are friends with each other.

Examples: Express in predicate logic using the following predicates.

$$B(x,y) = x\ is\ a\ brother\ of\ y, \qquad S(x,y) = x\ is\ a\ sibling\ of\ y$$
$$L(x,y) = x\ loves\ y$$

1. "Brothers are siblings."

   $\forall x \forall y\, (B(x,y) \rightarrow S(x,y))$

2. "Everybody loves somebody."

   $\forall x \exists y\, L(x,y)$

3. "There is someone who is loved by everyone."

   $\exists y \forall x\, L(x,y)$

4. "There is someone who loves someone."

   $\exists x \exists y\, L(x,y)$

5. "Everyone loves himself"

   $\forall x\, L(x,x)$

Negating Nested Quantifiers

- Use De Morgan's Laws to move the negation as far inwards as possible.

Example: Express the negation of the statement $\forall x \exists y (xy = 1)$ so that no negation precedes a quantifier.

$$\neg \forall x \exists y (xy = 1)$$
$$\equiv \exists x \neg \exists y (xy = 1)$$
$$\equiv \exists x \forall y \neg (xy = 1)$$
$$\equiv \exists x \forall y (xy \neq 1)$$