Discrete Mathematics
Proof Methods and Strategy

Exhaustive Proof

Some theorems can be proven by examining a relatively small number of examples. Such proofs are called exhaustive proofs (we just exhaust all the possibilities).

Example: Prove that there are NO positive perfect cubes less than 1000 that are the sum of the cubes of two positive integers.

Peoples can carry out exhaustive proofs only when it is necessary to check only a relatively small number of instances of a statement. Computers do better, but still there are limitations.

Proof by Cases

A proof by cases must cover all possible cases that arise in a theorem.

Example: Use a proof by cases to show that $min(a, min(b, c)) = min(min(a, b), c)$, whenever a, b, and c are real numbers.

Proof:

Case 1: a is the smallest or tied for the smallest

Case 2: b is the smallest or tied for the smallest

Case 3: c is the smallest or tied for the smallest

Common errors with exhaustive proofs and proofs by cases

We must consider all possible cases. No matter how many examples are considered, a theorem is NOT proved unless every possible example is covered.

Example: Conjecture: "If x is a real number, then x^2 is a positive real number."

Proof by cases (???):

Case 1: Let x be a positive real number. Then x^2 is positive since positive times positive is positive.

Case 2: Let x be a negative real number. Then x^2 is positive since negative times negative is positive.

Mistake:

Existence Proofs

Definition: A proof of a proposition of the form $\exists x P(x)$ is called an existence proof.

There are two types of existence proofs.

1. **Constructive**
   The proof is given by finding an element such that $P(a)$ is true.

2. **Nonconstructive**
   Someone shows that an element a such that $P(a)$ is true must exist but does not tell us what that element is.
   One method that could be used here is a proof by contradiction. We show that the negation of an existence quantifier implies a contradiction.

Example: Prove that there is a positive integer that can be written as the sum of squares of positive integers in two different ways.

Constructive proof:

Example: Theorem: Let $1, 2, \dots, n$ be natural numbers and k be their arithmetic mean (average), $k = \frac{1+2+\dots+n}{n}$. There exists a number m (among $1, 2, \dots n$) such that $m \geq k$.

Nonconstructive existence proof by contradiction:

Uniqueness Proofs

Some theorems assert the existence of a unique element with a particular property. In other words, these theorems assert there is exactly one element with this property.

To prove a statement of this type we need to show that an element with this property exists and no other element has this property.

The two parts of a uniqueness proof are:

1. ***Existence:*** We show that an element with a desired property exists.
2. ***Uniqueness:*** We show that if x and y both have the desired property, then $x = y$.

Example: Show that if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Proof:

1. ***Existence***

2. ***Uniqueness***