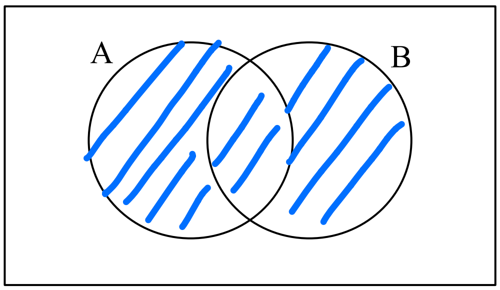
Discrete Mathematics
Set Operations

Definition: Let A and B be sets. The *union* of the sets A and B, denoted $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

Try this one: Write the union of A and B in the set builder notation.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$



Venn diagram for $A \cup B$

Definition: Let A and B be sets. The *intersection* of the sets A and B, denoted, $A \cap B$, is the set containing those elements in both A and B.
Two sets are called *disjoint* if their intersection is empty.

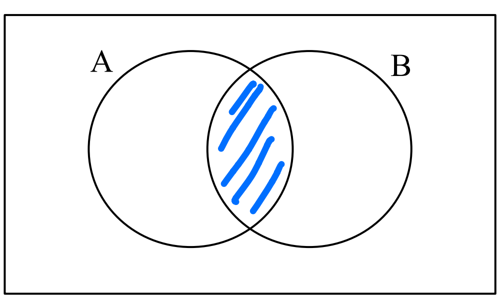
Try this one: Write the intersection of A and B in the set builder notation.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Example 1:

a. Find $\{5, 7, 19\} \cup \{1, 2, 3\}$. $= \{1, 2, 3, 5, 7, 19\}$

b. Find $\{5, 7, 19\} \cap \{1, 2, 3\}$. $= \emptyset$ $(\{\})$



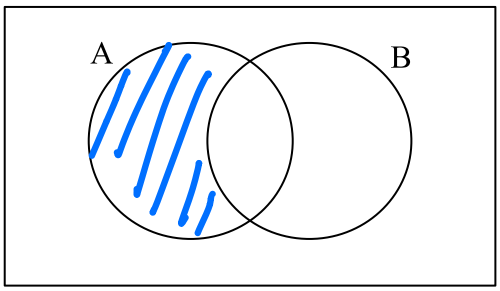
Venn diagram for $A \cap B$

$A \setminus B$

Definition: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

Try this one: Write the difference of A and B in the set builder notation.

$A - B = \{x \mid x \in A \wedge x \notin B\}$



Venn diagram for $A - B$

Example 2: Find $\{5, 7, 19\} - \{1, 2, 3\}$ $= \{5, 9, 19\}$

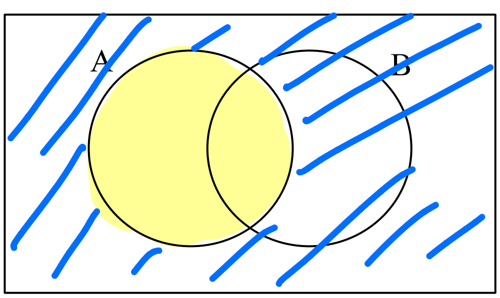
$\{2, 3\} - \{2, 3\} = \{\}$

A^c

Definition: Let U be the universal set. The complement of the set A, denoted by $\bar{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

Try this one: Write $\bar{A}$, in the set builder notation.

$\bar{A} = \{x \mid x \notin A\}$



Venn diagram for $\bar{A}$

Example 3: Let A be the set of positive natural numbers and the universal set is the set of natural numbers. Write $\bar{A}$.

$\bar{A} = \{0\}$ $\quad 0, 1, 2, \ldots$

The cardinality of the union of two sets

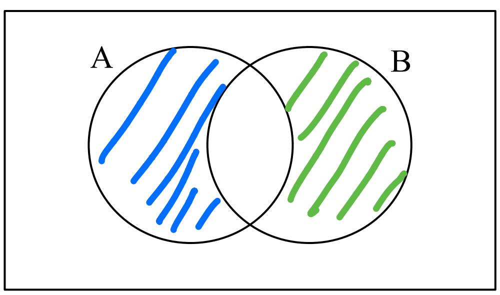
$|A \cup B| = |A| + |B| - |A \cap B|$ Inclusion-exclusion principle

Example 4: Find $|A \cup B|$ if $A = \{1,2,3,4,5\}$ and $B = \{2, 3, 4, 5,6,7,8\}$.

$|A \cup B| = \underline{5} + \underline{7} - 4 = 8$

Definition: Let A and B be sets. The *symmetric difference* of A and B, denoted by $A \oplus B$ is the set $(A - B) \cup (B - A)$.

$A \oplus B = (A \cup B) - (A \cap B)$



Venn diagram for symmetric difference

Try this one: $U = \{x \in \mathbb{N} \mid x \leq 10\}$, $A = \{1,2,3\}$, and $B = \{3, 4, 5\}$. Find $A \oplus B$.

$A \oplus B = \{1, 2, 4, 5\}$

Set Identities

- Identity laws: $A \cup \emptyset = A$; $A \cap U = A$
- Domination laws: $A \cup U = U$; $A \cap \emptyset = \emptyset$
- Idempotent laws: $A \cup A = A$; $A \cap A = A$
- Complementation law: $\overline{(\bar{A})} = A$
- Commutative laws: $A \cup B = B \cup A$; $A \cap B = B \cap A$
- Associative laws: $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive laws: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- De Morgan's laws: $\overline{A \cup B} = \bar{A} \cap \bar{B}$; $\overline{A \cap B} = \bar{A} \cup \bar{B}$
- Absorption laws: $A \cup (A \cap B) = A$; $A \cap (A \cup B) = A$
- Complement laws: $A \cup \bar{A} = U$; $A \cap \bar{A} = \emptyset$

1. $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$
2. $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

Proving set identities

$A = B$ iff $A \subseteq B \wedge B \subseteq A$

Different ways to prove set identities

1) • Prove that each set (side of the identity) is a subset of the other.

2) • Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

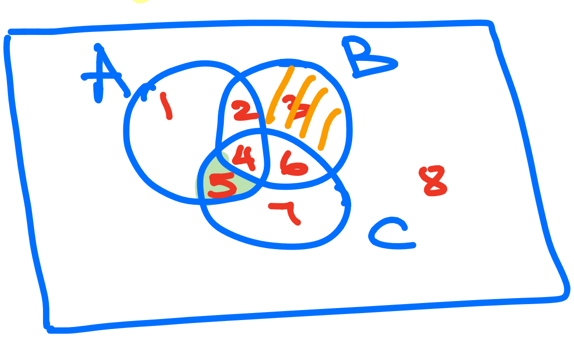
2) **Example 5:** Construct a membership table to show $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

A	B	C	$B \cap C$	LHS $A \cup (B \cap C)$	$A \cup B$	$A \cup C$	RHS
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

$x \notin A$
$x \in B$
$x \notin C$

$x \in A$
$x \notin B$
$x \in C$

same



1) **Example 6:** Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ by showing that RHS set and LHS set are subsets of each other.

① $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

LHS $\subseteq$ RHS

Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$ (or both).

If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$.

Hence $x \in (A \cup B) \cap (A \cup C)$.

If $x \in B \cap C$, then $x \in B$ <u>and</u> $x \in C$.

Then $x \in A \cup B$ <u>and</u> $x \in A \cup C$.

Hence $x \in (A \cup B) \cap (A \cup C)$.

② $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

RHS $\subseteq$ LHS

Let $x \in (A \cup B) \cap (A \cup C)$, then

$x \in A \cup B$ <u>and</u> $x \in A \cup C$.

Then $x \in A$ or $x \in B$ and $x \in C$ (or both)

If $x \in A$, then $x \in A \cup (B \cap C)$.

If $x \in B$ <u>and</u> $x \in C$, then $x \in B \cap C$.

Hence $A \cup (B \cap C)$.

QED.

Definition: Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

$(a,b) \neq (b,a)$

Try this: Write $A \times B$ in the set builder notation.

$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$

Example 7: Let $A = \{1,2,3\}$ and $B = \{a, b\}$. Find $A \times B$ and $B \times A$.

$A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$B \times A = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

In general, $A \times B \neq B \times A$

$A \times B = B \times A$, when $A = B$

Example 8: Find A^3 if $A = \{a\}$.

$A^2 = \{(a,a)\}$

$A = \{a\}$

$A^3 = A \times A \times A = \{(a,a,a)\}$

$A^2 \times A = \{((a,a),a)\}$

Example 9: How many different elements does $A \times B$ have if A has m elements and B has n elements?

$m \times n$

Example 10: Suppose $A \times B = \emptyset$, where A and B are sets. What can you conclude?

$|A \times B| = \emptyset$, then $A = \emptyset$ or $B = \emptyset$ or both

$A = \{a,b,c\}$ $\quad B = \emptyset$

$A \times B = \{(\_,\_), \ldots\} = \emptyset$

$(a,b) \neq \{a,b\}$

(a,b): ordered pair

$\{a,b\}$: set containing 2 elements a and b

$(\underset{1}{\underline{a}}, \underset{2}{\underline{b}})$

$\boxed{a \quad b}$

$\{a,b\} = \{b,a\}$

$(a,b) \neq (b,a)$