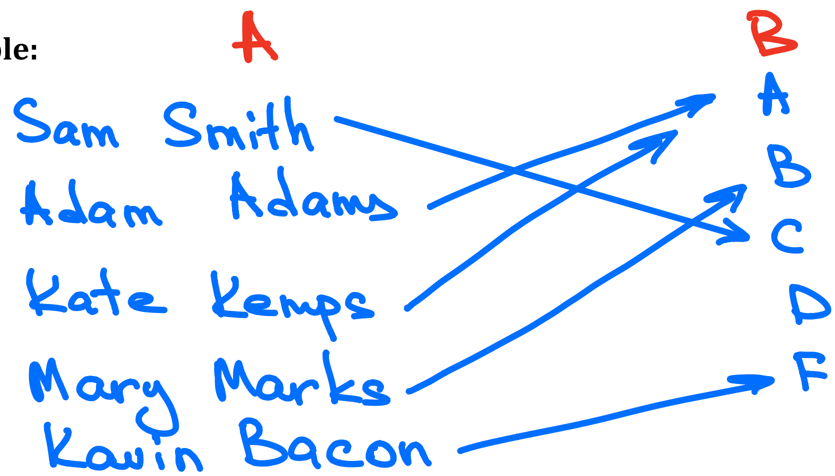
Discrete Mathematics
Functions

Definition: A *function f* from a set A to a set B, denoted $f: A \to B$ is a well-defined rule that assigns each element of A to exactly one element of B.

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

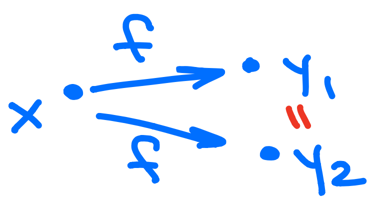
Example:



Recall: Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

We can define a function $f: A \to B$ as a subset of the Cartesian product $A \times B$.

$$\forall x\, [x \in A \to \exists y\, [y \in B \wedge (x, y) \in f]]$$
$$\wedge\ \forall x, y_1, y_2\, [[(x, y_1) \in f \wedge (x, y_2) \in f] \to y_1 = y_2]$$

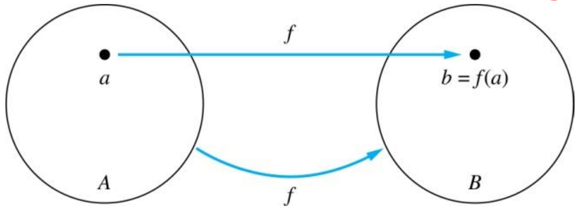


$\frac{0}{0}$ $\lim_{x \to 2} \frac{x^2-4}{x-2}$ $D: \mathbb{R}\setminus\{2\}$

$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$

$= \lim_{x \to 2} (x+2) = 4$ $D: \mathbb{R}$

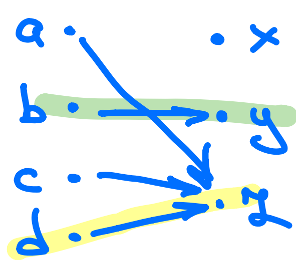
Given a function $f: A \to B$:



- We say f *maps* A to B or f is a *mapping* from A to B.
- A is called the *domain* of f.
- B is called the *codomain* of f.
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.
- The range of f is the set of all images of points in **A** under f. We denote it by $f(\mathbf{A})$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Example: Let $f: A \to B$, where $A = \{a, b, c, d\}$, $B = \{x, y, z\}$, and $f(a) = z, f(b) = y, f(c) = z, f(d) = z$.

$f: A \to B$



a. $f(a) = ?$ z

b. The image of d is? z

c. The domain of f is? $\{a, b, c, d\}$

d. The codomain of f is? $\{x, y, z\}$

e. The preimage of y is? b

f. $f(A) = ?$ $\{y, z\}$

g. The preimage(s) of z is (are)?

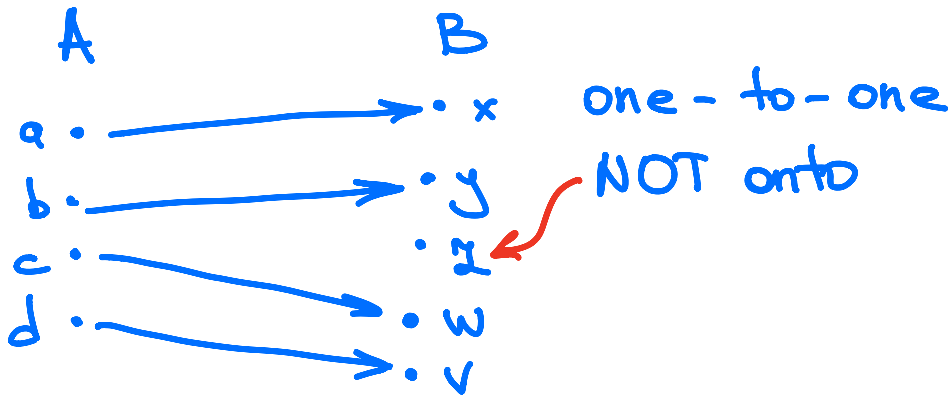
$\{a, c, d\}$

h. The range of f is?

$\{y, z\}$

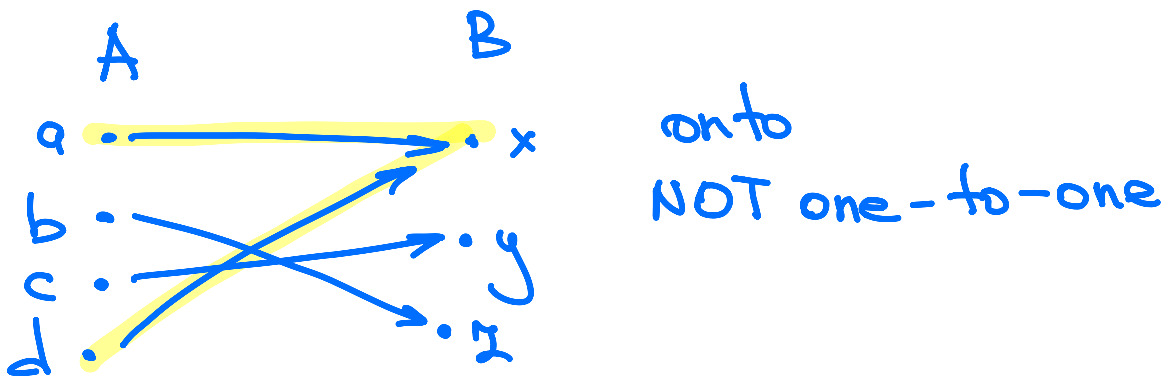
Definition: A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.

Example:



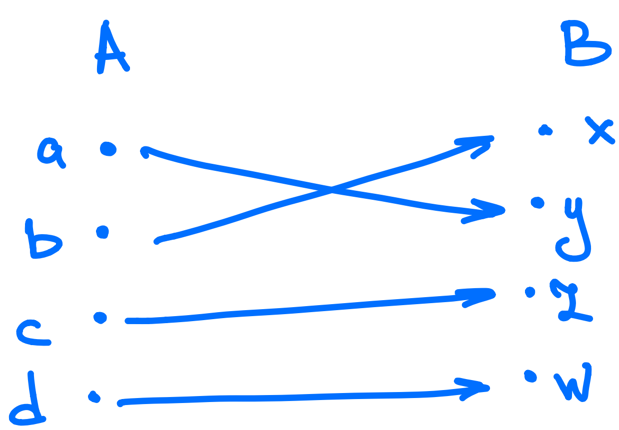
Definition: A function f from A to B is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function *f* is called a *surjection* if it is *onto*.

Example:



Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).

Example:



Suppose that $f: A \to B$.

To show that f is injective: Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

To show that f is *not* injective: Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective: Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is *not* surjective: Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Example: Let f be the function from $\{a, b, c, d\}$ to $\{1,2,3\}$ defined by $f(a) = 3, f(b) = 2, f(c) = 1$, and $f(d) = 3$. Is f an onto function?

YES

Example: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

No, no such integer that $x^2 = 3$.

Example: Determine if each function is a bijection. $\mathbb{R}$

a. $f(x) = 2x + 1$

✓ Injective: If $f(a) = f(b)$, then $a = b$.

$$\frac{2a}{2} + \cancel{1} = \frac{2b}{2} + \cancel{1} \rightarrow a = b$$

✓ Surjective: Let $y \in \mathbb{R}$, then $2x + 1 = y$

$$x = \frac{y-1}{2} \in \mathbb{R}$$

Hence $f(x)$ is a bijection.

b. $f(x) = x^2 + 1$ $\quad \mathbb{R} \to \mathbb{R}$

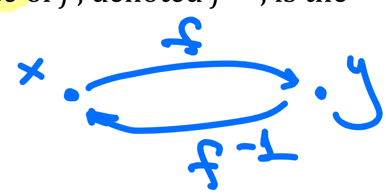
NOT Injective: $f(a) = f(b) \to a = b$

Take $x = 2$ and $y = -2$.

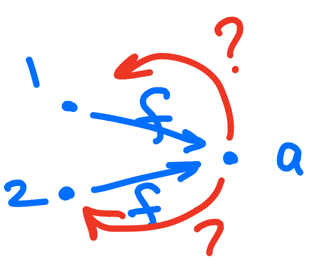
$f(2) = 5$
$\quad\;\; \|$
$f(-2) = 5$

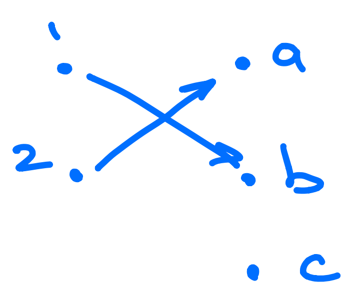
Hence, $f(x)$ is NOT bijective.

Definition: Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as $f^{-1}(y) = x$ iff $f(x) = y$.



No inverse exists unless f is a bijection. Why?





$f(1) = a$

$f(2) = a$

$f^{-1}(a) = ?$

$f^{-1}(a) = 2$

$f^{-1}(b) = 1$

$f^{-1}(c) = ?$

Example: Let f be the function from $\{a, b, c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

$f^{-1}(1) = c$

$f^{-1}(2) = a$

$f^{-1}(3) = b$

$\mathbb{R} \to \mathbb{R}$

Example: Let $f: \mathbb{Z} \to \mathbb{Z}$ be such that $f(x) = 2x + 1$. Is f invertible, and if so, what is its inverse?

No, not onto

Take $y = 2$

$2x + 1 = 2$

$2x = 1$

$x = 1/2 \notin \mathbb{Z}$

$y = 2x + 1$

$x = \frac{y-1}{2} \in \mathbb{R}$

YES

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be such that $f(x) = |x|$. Is f invertible, and if so, what is its inverse?

Not one-to-one

$x = 1 \qquad f(1) = 1$

$x = -1 \qquad f(-1) = 1$

Definition: Let $f: B \to C$, $g: A \to B$. The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by $(f \circ g)(x) = f(g(x))$.

Example: $f(x) = x^2$ and $g(x) = 2x + 1$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$(f \circ g)(x) = f(g(x)) = (2x+1)^2 = \underline{4x^2+4x+1}$

$(g \circ f)(x) = g(f(x)) = \underline{2x^2+1}$

$(f \circ g)(x) \neq (g \circ f)(x)$

If $f(x) = g^{-1}(x)$, then $(f \circ g)(x) = (g \circ f)(x) = x$.

Example: Let *g* be the function from the set {*a, b, c*} to itself such that *g*(*a*) = *b*, *g*(*b*) = *c*, and *g*(*c*) = *a*. Let *f* be the function from the set {*a, b, c*} to the set {1,2,3} such that *f*(*a*) = 3, *f*(*b*) = 2, and *f*(*c*) = 1.

What is the composition of *f* and *g*, and what is the composition of *g* and *f*.

$(f \circ g)(x) = f(g(x))$

$(f \circ g)(a) = 2$

$(f \circ g)(b) = 1$

$(f \circ g)(c) = 3$

$(g \circ f)(x) = g(f(x))$

DNE

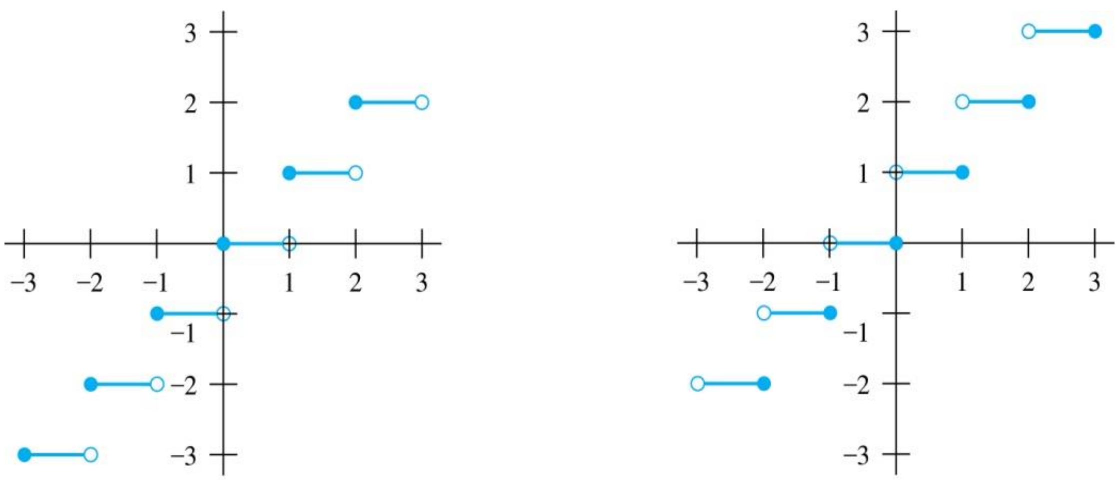
range of $f \neq$ domain of g

Some important functions

- The *floor* function, denoted $\lfloor x \rfloor$ is the largest integer less than or equal to x.
- The *ceiling* function, denoted $\lceil x \rceil$is the smallest integer greater than or equal to x.

Example: $\lfloor 3.5 \rfloor = 3$ $\lceil 3.5 \rceil = 4$

$\lfloor 2.999 \rfloor = 2$ $\lceil 2.0001 \rceil = 3$



(a) $y = \lfloor x \rfloor$ (b) $y = \lceil x \rceil$

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

- Factorial function

Definition: $f: \mathbb{N} \rightarrow \mathbb{Z}^{+}$**,** denoted by $f(n) = n!$ is the product of the first n positive integers.
$f(n) = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. $f(0) = 0! = 1$

Example: Find f(2), f(3), f(4).

$2! = 1 \cdot 2 = 2$

$3! = 1 \cdot 2 \cdot 3 = 6$

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$n! = n(n-1)!$