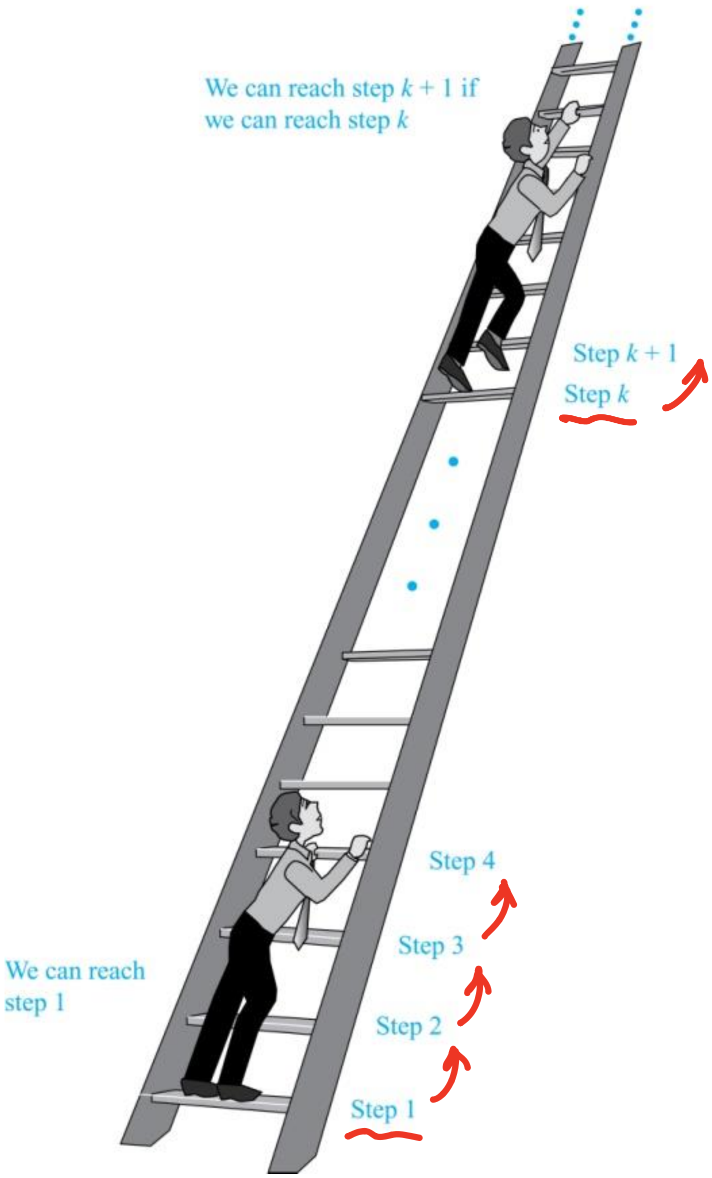
MATH 3336 – Discrete Mathematics
Mathematical Induction (5.1)



(from Discrete Mathematics and Its Applications by K. Rosen)

Suppose we have an infinite ladder:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

$n \geq 1$

Principle of Mathematical Induction: To prove that $P(n)$ is true for all positive integers n, we complete these steps:

- *Basis Step*: Show that $P(1)$ is true.
- *Inductive Step*: Show that $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

To complete the inductive step, assuming the *inductive hypothesis* that $P(k)$ holds for an arbitrary integer k, show that must $P(k+1)$ be true.

Important points about Mathematical Induction

- Mathematical induction can be expressed as the rule of inference where the domain is the set of positive integers.

$$\left(P(1) \wedge \forall k\,(P(k) \rightarrow P(k+1))\right) \rightarrow \forall n\, P(n)$$

- In a proof by mathematical induction, we do not assume that $P(k)$ is true for all positive integers! We show that if we assume that if $P(k)$ is true, then $P(k+1)$ must also be true.
- Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a starting point b where b is an integer. We will see examples of this soon.
- Mathematical Induction cannot be used to find new theorems and does not give insights on why a theorem works.

Example: Show that if n is a positive integer, then $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Basis step: $P(1)$ $\quad 1 = \frac{1(1+1)}{2} \quad 1 = 1$ true

Inductive step: $P(k) \rightarrow P(k+1)$

If $1+2+\ldots+k = \frac{k(k+1)}{2}$, then $1+2+\ldots+k+k+1 = \frac{(k+1)(k+2)}{2}$.

Ind. Hyp

$$\underbrace{1+2+\ldots+k}+k+1 = \frac{k(k+1)}{2} + k+1 = (k+1)\left[\frac{k}{2}+1\right]$$

$$= (k+1)\left(\frac{k+2}{2}\right) = \frac{(k+1)(k+2)}{2}.$$

Conclusion: By the principle of induction, the statement is true for all $n \in \mathbb{Z}^+$.

Example: Conjecture a formula for the sum of the first n positive odd integers. Then prove the conjecture using mathematical induction.

$1 = 1 = 1^2$

$1+3 = 4 = 2^2$

$1+3+5 = 9 = 3^2$

$1+3+5+7 = 16 = 4^2$

Conjecture: $1+3+5+\ldots+(2n-1) = n^2$

Basis step: $P(1)$ $\quad 1 = 1^2$ true

Inductive step: $P(k) \rightarrow P(k+1)$

If $1+3+\ldots+(2k-1) = k^2$, then $1+3+\ldots+(2k-1)+(2k+1) = (k+1)^2$.

$$\underbrace{1+3+5+\ldots+(2k-1)}+(2k+1) \overset{\text{Ind Hyp}}{=} k^2+2k+1$$

$$= (k+1)^2.$$

Conclusion: By the principle of induction, the statement is true for all odd positive n.

$2^1 \not< 1!$ $\quad 2^2 \not< 2!$ $\quad 2^3 \not< 3!$

Example: Use mathematical induction to prove $2^n < n!$ For every integer n with $n \geq 4$.

Basis step: $P(4)$ $\quad 2^4 < 4! \quad 16 < 24$ true

Inductive step: $P(k) \rightarrow P(k+1)$

If $2^k < k!$, then $\underbrace{2^{k+1}}_{LHS} < \underbrace{(k+1)!}_{RHS}$.

$$\underbrace{2^{k+1}}_{LHS} = 2^k \cdot 2 \overset{\text{Ind.Hyp}}{<} k! \cdot 2 \overset{2<k+1}{<} k!\,(k+1) = \underbrace{(k+1)!}_{RHS}$$

Conclusion: By the principle of induction, the statement is true for all integer $n \geq 4$.

Example: Prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Basis step: $P(1)$ $3 \mid 1^3 - 1$ $3 \mid 0$ true

Inductive step: $P(k) \rightarrow P(k+1)$

If $3 \mid (k^3 - k)$, then $3 \mid (k+1)^3 - (k+1)$.

$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + \cancel{1} - k - \cancel{1}$

$= (k^3 - k) + 3(k^2 + k)$

$3 \mid (k^3 - k)$ Inductive Hyp.

$3 \mid \underline{\underline{3}}(k^2 + k)$

$3 \mid (k^3 - k) + 3(k^2 + k)$

Conclusion: By the principle of induction, the statement is true for all $n \in \mathbb{Z}^+$.

Example: Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

Basis step: $P(0)$ $\quad 57 \mid (7^2 + 8) \qquad 57 \mid 57$ true

Inductive step: $P(k) \rightarrow P(k+1)$

If $57 \mid 7^{k+2} + 8^{2k+1}$ then $57 \mid 7^{k+3} + 8^{2k+3}$.

$$7^{k+3} + 8^{2k+3} = 7(7^{k+2}) + 8^2(8^{2k+1})$$

$$= 7(7^{k+2}) + \underset{(7+57)}{\underset{\shortparallel}{64}}(8^{2k+1})$$

$$= 7(7^{k+2}) + (7+57)(8^{2k+1})$$

$$= 7(7^{k+2}) + 7(8^{2k+1}) + 57(8^{2k+1})$$

$$= 7(\underbrace{7^{k+2} + 8^{2k+1}}_{\text{by I.H.}}) + 57(8^{2k+1})$$

$57 \mid$ by I.H. $\qquad 57 \mid 57(8^{2k+1})$

Conclusion: By the principle of induction, the statement is true for all n ~~≥ 3?~~. nonnegative

Hence $57 \mid 7(7^{k+2} + 8^{2k+1}) + 57(8^{2k+1})$

Example: Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.

Basis step: $P(0)$ $|S| = 0$ $S = \emptyset$ $\emptyset$ set has $2^0 = 1$ 1 subset $\emptyset \subseteq \emptyset$ true

Inductive step: $P(k) \rightarrow P(k+1)$

Let A be a set with k elements.
Assume A has 2^k subsets.
Then $|B| = k+1$ has 2^{k+1} subsets.

$B = A + \{a\}$

Subsets of B: 1. Subsets of A
There are 2^k of these (I.H)
2. $S \cup \{a\}$ for $S \subseteq A$. There are 2^k of this type.
Total # of subsets: $2^k + 2^k = 2^{k+1}$

Conclusion: By the principle of induction, the statement is true for all nonnegative integers.

$A = \{1,2\}$
$B = \{1,2,3\}$

$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$\mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\},$
+3 +3 +3 +3
$\{1,3\}, \{2,3\}, \{1,2,3\}\}$