MATH 3336 – Discrete Mathematics
Strong Induction (5.2)

Principle of Strong Mathematical Induction: To prove that $P(n)$ is true for all positive integers n, we complete these steps:

- *Basis Step*: Show that $P(1)$ is true.
- *Inductive Step*: Show the conditional statement $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

- Inductive Hypothesis includes ALL k statements $P(1), P(2), \ldots, P(k)$.

Example: Every positive integer greater than 1 can be expressed as a product of primes.

Proof:

Basis step: $P(2)$ 2 can be expressed as a product of primes. $2 = 2$

Inductive step: $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$

Assume all positive integers greater than 1 and less that or equal to k can be expressed as a product of primes, then k+1 can be expressed as a product of primes.

k+1

case 1: k+1 is prime. Then $k+1 = k+1$.

case 2: k+1 is composite
Then $k+1 = a \cdot b$ $\quad 2 \leq a \leq k,\ 2 \leq b \leq k$

By Ind. Hyp. a and b can be expressed as product of primes. Hence, k+1 can be written as a product of primes.

Conclusion: By the principle of ***strong*** mathematical induction, the statement is true for all integers greater than 1.

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Basis step: $P(12)$ Use three 4-cent stamps.

Inductive step: If $P(j)$ for $12 \le j \le k$, then $P(k+1)$.

One can assume $P(k-3)$ is true (I.H.)

We can form a postage of $(k-3)$ ¢ with 4-cent and 5-cent stamps.

Add one 4-cent stamp $(k-3+4 = k+1)$ to form a postage of $(k+1)$ ¢.

Conclusion: By the principle of ***strong*** mathematical induction, the statement is true for all integers greater than 12.

Example: Let $f_n = f_{n-1} + f_{n-2}$, $f_0 = 1$, and $f_1 = 1$. Show that $f_n \le 2^n$ for all positive n.

Proof:

Basis step: $P(1)$

$f_1 \le 2^1 \quad 1 \le 2$ true

$f_2 \le 2^2 \quad 1 \le 4$ true

Inductive step: If $P(j)$ for $1 \le j \le k$, then $P(k+1)$.

$f_j \le 2^j$ for $1 \le j \le k$

$$f_{k+1} = f_k + f_{k-1} \underset{\text{Ind. Hyp}}{\le} 2^k + 2^{k-1}$$

$$= 2^{k-1}(2+1)$$

$$= 2^{k-1} \cdot 3 < 2^{k-1} \cdot 4 =$$

$$= 2^{k-1} \cdot 2^2 = 2^{k+1}$$

Conclusion: By the principle of ***strong*** mathematical induction, the statement is true for all positive integers.