MATH 3336 – Discrete Mathematics
The Basics of Counting (6.1)

Basic Counting Principles: The Product Rule

The Product Rule: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 n_2$ ways to do the procedure.

Example: How many bit strings of length seven are there?

0, 1

$\underset{\uparrow}{\_}\ \underset{\uparrow\,2}{\_}\ \underset{\uparrow\,2}{\_}\ \ldots\ \_\ \_\ \_\ \underset{\uparrow\,2}{\_}$ $\boxed{2^7}$

2 choices
0 or 1

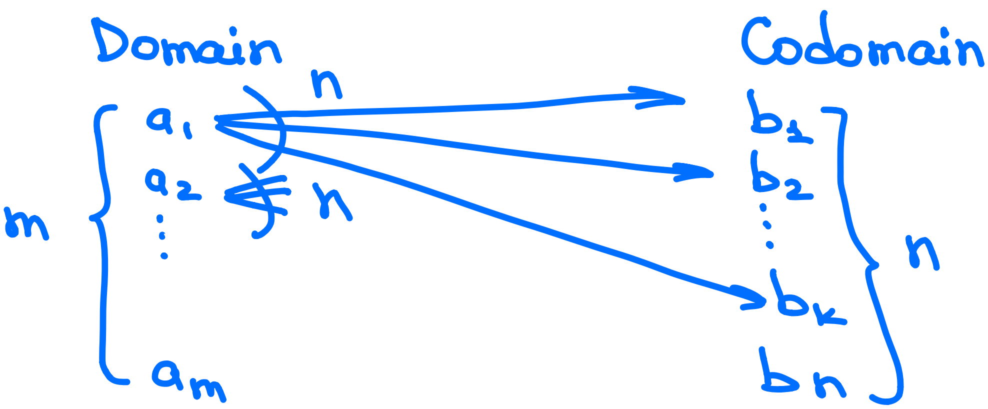
Example: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

letters: $\_\ \_\ \_$ — 26 choices for each

digits: $\_\ \_\ \_$ — 10 choices for each

$\boxed{26^3 \cdot 10^3}$ $= 17,576,000$

Example (Counting Functions): How many functions are there from a set with m elements to a set with n elements?



$$\underbrace{n \cdot n \cdot \ldots n}_{m} = n^m$$

Example (Counting One-to-One Functions): How many one-to-one functions are there from a set with m elements to one with n elements?

$f(a) = f(b) \rightarrow a = b$

Domain

$$m\begin{cases} a_1 \leftarrow n \\ a_2 \leftarrow n-1 \\ \vdots \\ a_m \end{cases}$$

Codomain

$$\left.\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix}\right\} n$$

$\#$ functions $= n(n-1)\dots(n-m+1)$.

Example (Counting Subsets of a Finite Set): Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$. *(In Section 5.1, mathematical induction was used to prove this same result.)*

Every subset of S can be written as a binary string of length $|S|$.

$\boxed{2^{|S|}}$

Ex $S = \{1,2,3,4\}$

$\emptyset \rightarrow 0000 \qquad \{1,4\} \rightarrow 1001$

Basic Counting Principles: The Sum Rule

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

$37 + 83 = 120$ choices

More Complex Counting Problems

Example: In a version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. (An alphanumeric character is either one of the 26 English letters or one of the 10 digits.) Moreover, a variable name must begin with a letter and must be different from the five strings that are reserved for programming use. How many different variable names are there in this version of BASIC?

$\underline{26}$

$\underline{26} \cdot \underline{36}$

$26 + 26 \cdot 36 - 5 = \boxed{957}$

Example (Counting Passwords): Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution:
Let P_6, P_7, and P_8 be the number of passwords of length 6, 7, and 8 correspondingly.

Total number of passwords = $P_6 + P_7 + P_8$

$P_6 = 36^6 - 26^6$

$P_7 = 36^7 - 26^7$

$P_8 = 36^8 - 26^8$

Total number of passwords = $36^6 + 36^7 + 36^8 - 26^6 - 26^7 - 26^8$

$= 2,684,483,063,360$

$|A \cup B| = |A| + |B| - |A \cap B|$

Subtraction Rule (Inclusion-Exclusion for Two Sets)

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Example (Counting Bit Strings): How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution:

Number of bit strings of length 8 that start with a 1 bit = 2^7

Number of bit strings of length 8 that end with bits 00 = 2^6

10010100

Number of bit strings of length 8 that start with a 1 bit and end with bits 00 = 2^5

Total number of bit strings of length 8 that either start with a 1 bit or end with the two bits 00 = $2^7 + 2^6 - 2^5 = 128 + 64 - 32 = 160$

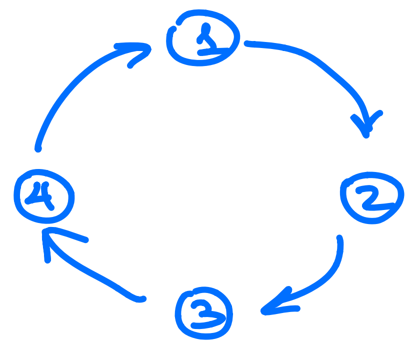
The Division Rule

Division Rule: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

Example: How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

4 3 2 1

4!

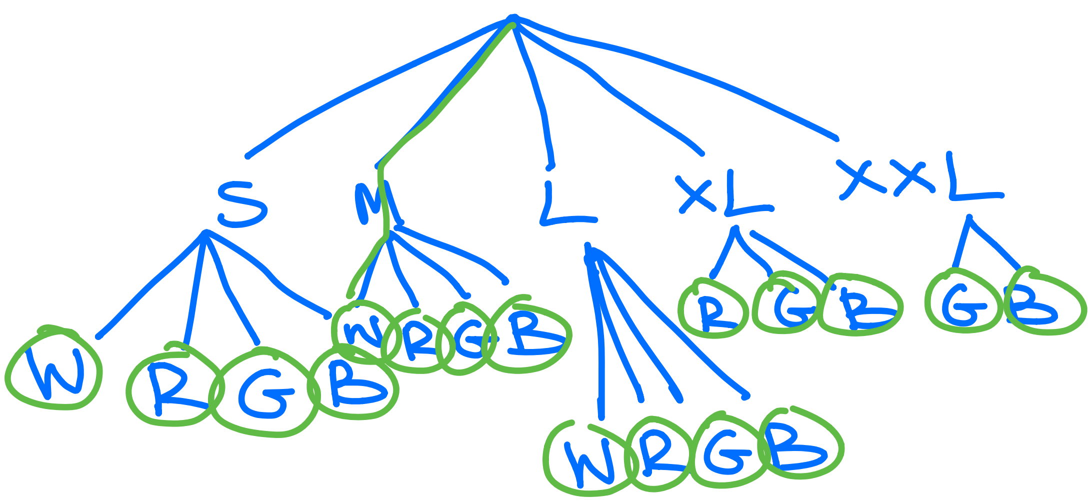


$\boxed{\frac{4!}{4}}$

Tree Diagrams

Tree Diagrams: We can solve many counting problems with *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.

Example: Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S, M, L, XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?



17 T-shirts