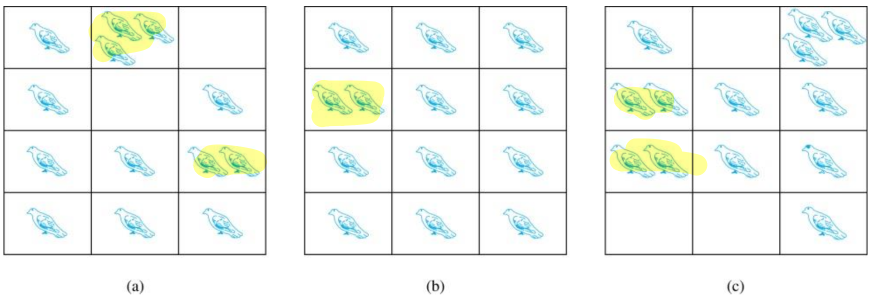
MATH 3336 – Discrete Mathematics
The Pigeonhole Principle (6.2)

The Pigeonhole Principle

If a flock of 13 pigeons roosts in a set of 12 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



(from Discrete Mathematics and Its Applications by K. Rosen)

The Pigeonhole Principle: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: Assume none of the k boxes contains more than one object.

Then the # of objects is at most k.

Contradicts having $(k+1)$ objects.

Q.E.D.

Example: A function f from a set with $k + 1$ elements to a set with k elements is not one-to-one.

- Create a box for each element in the codomain. We have k boxes.
- Put $(k+1)$ elements from the domain in k boxes. By Pigeonhole, one box has 2 or more elements.

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example: There are 10 black socks and 10 white socks in a drawer. How many socks you need to pick (without looking) if you want to wear a pair of socks of the same color?

Black White $\boxed{3}$

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: Suppose none of the boxes contain more than $\lceil \frac{N}{k} \rceil - 1$ objects.

Total # of objects is at most

$$k\Big(\underbrace{\Big\lceil \frac{N}{k} \Big\rceil}_{\lceil \frac{N}{k} \rceil < \frac{N}{k} + 1} - 1\Big) < k\Big(\frac{N}{k} + \cancel{1} - \cancel{1}\Big) = N$$

Contradiction.

Q.E.D.

Example: Among 100 people there are at least $\left\lceil \frac{100}{12} \right\rceil = 9$ who were born in the same month.

Example:

a. How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
b. How many must be selected to guarantee that at least three hearts are selected?

(a) Assume 4 boxes (4 suits)

$\left\lceil \frac{N}{4} \right\rceil \geq 3 \qquad N = 4(2) + 1 = 9$

(b)

52

♥ 13 NOT ♥ 39

$39 + 3 = 42$