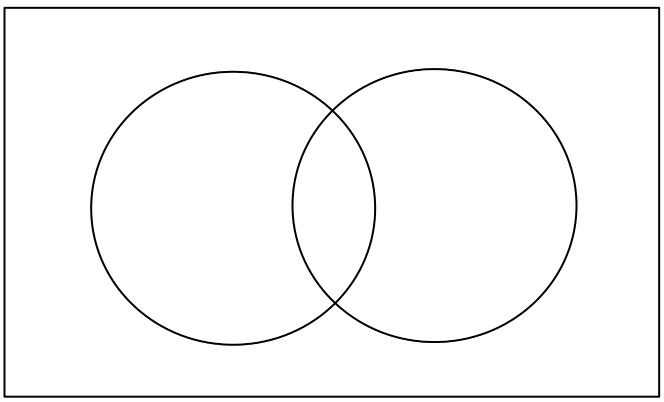
MATH 3336 – Discrete Mathematics
Inclusion-Exclusion Principle (8.5, 8.6)

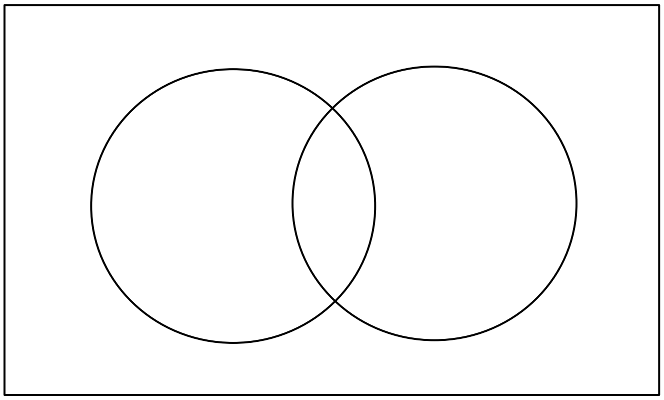
Earlier we developed the following formula for the number of elements in the union of two finite sets: $|A \cup B| = |A| + |B| - |A \cap B|$
We will generalize this formula to finite sets of any size.

Two Finite Sets

Example: In a discrete mathematics class, every student is a major in computer science or mathematics or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in the class?



Example: In a survey of 374 coffee drinkers, it was found that 64 take only sugar, 82 take only cream, and 65 do not take sugar nor cream with their coffee. How many take:



a. sugar and cream with their coffee?

b. cream?

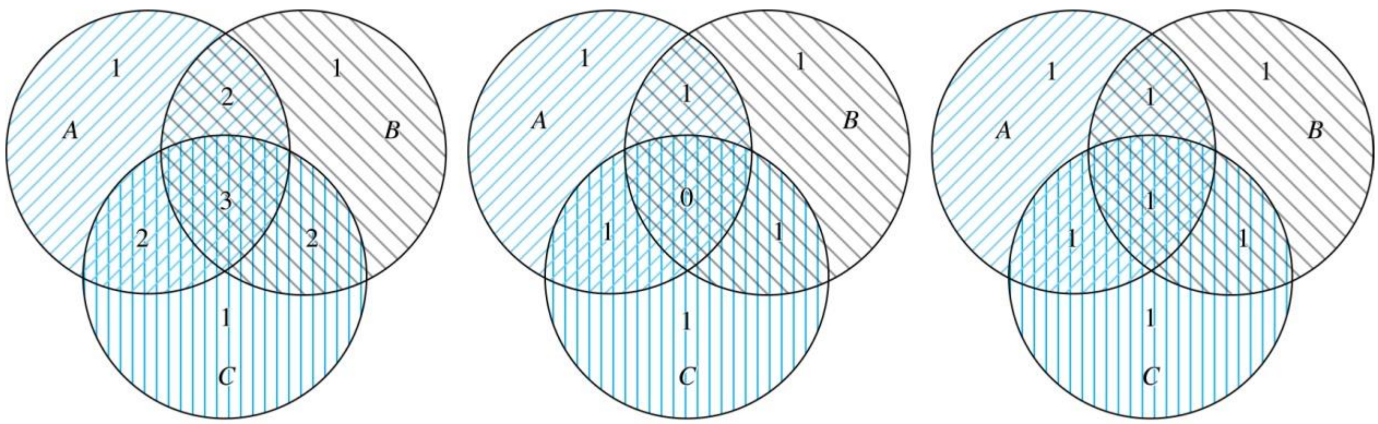
c. neither sugar nor cream?

d. exactly one of these two additions?

e. at least one of these two additions?

Three Finite Sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|+|A \cap B \cap C|$

(from Discrete Mathematics and Its Applications by K. Rosen)

Example: A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken a course in at least one of Spanish, French and Russian, how many students have taken a course in all 3 languages.

Given:

$|S| =$

$|F| =$

$|R| =$

$|S \cap F| =$

$|S \cap R| =$

$|F \cap R| =$

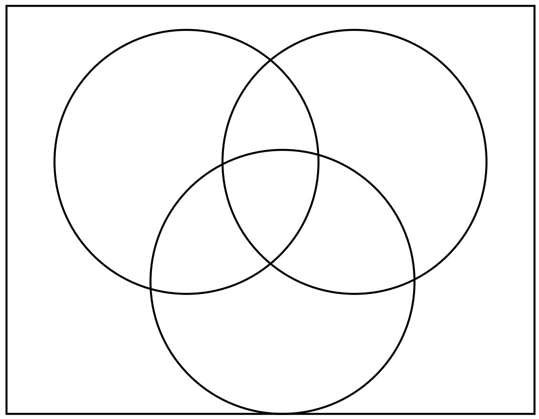
$|S \cup F \cup R| =$

Find: $|S \cap F \cap R|$

Solution:

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

Example: Sixty-seven students were asked about which subject they enjoy most. The survey revealed that 33 enjoy Math, 45 enjoy Science, 40 enjoy English, 17 enjoy Science and English only, 16 enjoy both Math and Science, 14 enjoy all three subjects, and 25 enjoy exactly two of the three subjects.



a. How many students surveyed enjoy Math and English?

b. How many students surveyed enjoy Science or English but not Math?

c. How many students surveyed enjoy at most one of the three subjects mentioned?

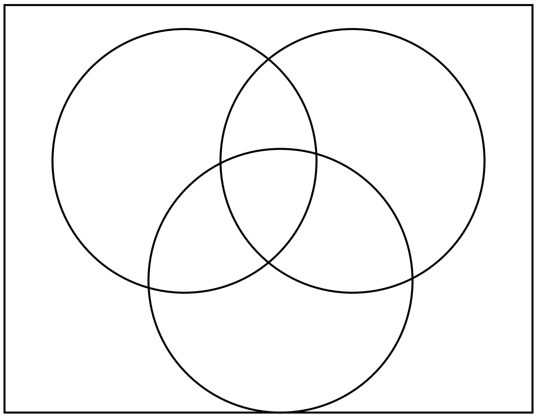
Theorem (The Principle of Inclusion-Exclusion): Let $A_1, A_2, \dots, A_n$ be finite sets. Then:

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i\le j\le n} |A_i \cap A_j| + \sum_{1\le i\le j\le k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1}|A_1 \cap A_2 \cap \dots \cap A_n|$$

Example: Find the number of positive integers ≤ 1000 that are multiples of at least one of 4, 6, and 15.

The Number of Onto Functions

Example: How many onto functions are there from a set with six elements to a set with three elements?



Theorem: Let m and n be positive integers with $m \geq n$. Then, there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1}C(n,n-1) \cdot 1^m$$

onto functions from a set with m elements to a set with n elements.

Derangements

Definition: A *derangement* is a permutation of objects that leaves no object in the original position.

Example:

a. The permutation of 21453 is a derangement of 12345.
b. 21543 Is not a derangement of 12345.

Theorem: The number of derangements of a set with n elements is

$$D_n = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

Example (University of South Carolina High School Math Contest, 1993): Suppose that 4 cards labeled 1 to 4 are placed randomly into 4 boxes also labeled 1 to 4, one card per box. What is the probability that no card is placed into a box having the same label as the card?

(A) 1/3 (B) 3/8 (C) 5/12 (D) 1/2 (E) 9/16