Math 3336 Test 1 Review Questions

How to study: Study the class notes, review homework and quiz problems, and try to do as many exercises as you can from the textbook. Note that answers are provided at the back of the book to all odd numbered problems.
Here I provide some examples for you. This is not a complete list, studying only these examples is not enough!

1. Determine whether this proposition is a tautology: $((p \to \neg q) \wedge q) \to \neg p$.

p	q	$\neg p$	$\neg q$	$p \to \neg q$	$(p \to \neg q) \wedge q$	$(p \to \neg q) \wedge q \to \neg p$
T	T	F	F	F	F	T
T	F	F	T	T	F	T
F	T	T	F	T	T	T
F	F	T	T	T	F	T

The proposition IS a tautology.

2. Construct a truth table for the following proposition: $(p \oplus q) \leftrightarrow (p \vee q)$.

p	q	$p \oplus q$	$p \vee q$	$(p \oplus q) \leftrightarrow (p \vee q)$
T	T	F	T	F
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

3. Determine whether the proposition is TRUE or FALSE.

$$\underbrace{If\ \overset{T}{1+1=2}\ or\ \overset{F}{1+1=3}}_{\text{True}},\ then\ \underbrace{\overset{F}{2+2=3}\ and\ \overset{T}{2+2=4}}_{\text{False}}.$$

$\boxed{\text{False}}$

4. Write the inverse, converse and contrapositive for the following statement.

I come to class (q) *whenever there is going to be a quiz* (p).

If p, then q
p only if q
p implies q
q follows from p
q when p (whenever)

If there is going to be a quiz, then I come to class.

Inverse ($\neg p \rightarrow \neg q$):

If there is not going to be a quiz, then I do not come to class.

Converse ($q \rightarrow p$):

If I come to class, then there is going to be a quiz.

Contrapositive ($\neg q \rightarrow \neg p$):

If I do not come to class, then there is not going to be a quiz.

5. Suppose $P(x, y)$ is a predicate and the domain for the variables x and y is $\{1,2,3\}$. Suppose $P(1,3), P(2,1), P(2,2), P(2,3), P(3,1)$, and $P(3,2)$ are true and if false otherwise. Determine whether the following statements are true.
    a. $\exists x \forall y\, P(x, y)$ True , $x = 2$
    b. $\forall x \exists y\, P(x, y)$ True

x \ y	1	2	3
1	F	F	T
2	T	T	T
3	T	T	F

6. Suppose the variable x represents students and y represents courses, and:
    $U(y)$: y is an upper-level course
    $M(y)$: y is a math course
    $F(x)$: x is a freshman
    $B(x)$: x is a full-time student
    $T(x, y)$: student x is taking course y

   Write the statement using these predicates and any needed quantifiers.
    a. Mark is taking Math 3336.
    b. All students are freshmen.
    c. Every freshman is a full-time student.
    d. No math course is upper-level.

a) $T(\text{Mark}, \text{Math } 3336)$

b) $\forall x\, F(x)$

c) $\forall x\, (F(x) \rightarrow B(x))$

d) $\forall y\, (M(y) \rightarrow \neg U(y))$

7. Express the negation of each statement so that all negation symbols immediately precede predicates.
   a. $\exists x \exists y (Q(x,y) \rightarrow Q(y,x))$
   b. $\exists x \exists y P(x,y) \wedge \forall x \forall y Q(x,y)$

a) $\neg(\exists x \exists y (Q(x,y) \rightarrow Q(y,x)))$

$\equiv \forall x \forall y \neg(\neg Q(x,y) \vee Q(y,x))$

$\equiv \forall x \forall y (Q(x,y) \wedge \neg Q(y,x))$

b) $\neg(\exists x \exists y P(x,y) \wedge \forall x \forall y Q(x,y))$

$\equiv \forall x \forall y \neg P(x,y) \vee \exists x \exists y \neg Q(x,y)$

8. Suppose you wish to prove a theorem of the form "*if p then q*".
   a. If you give a direct proof, what do you assume and what do you prove?
   b. If you give a proof by contraposition, what do you assume and what do you prove?
   c. If you give a proof by contradiction, what do you assume and what do you prove?

a) Assume p, show q.

b) Assume $\neg q$, show $\neg p$.

c) Assume $p \wedge \neg q$, show this leads to a contradiction.

9. Consider the following theorem: "*If x and y are odd integers, then $x + y$ is even*".
   a. Give a direct proof of this theorem.
   b. Give a proof by contradiction.

a) Let x and y be odd integers.
Then $x = 2k+1$ for some integer k,
and $y = 2\ell+1$ for some integer ℓ.
Then $x+y = 2k+1+2\ell+1 = 2(k+\ell+1)$,
which is even.
Q. E. D.

b) Suppose $x = 2k+1$, $y = 2\ell+1$, but
$x+y = 2m+1$.
Then $(2k+1)+(2\ell+1) = 2m+1$.
Hence $2(k+\ell+1) = 2m+1$

$$\underset{\text{even}}{2(k+\ell+1)} = \underset{\text{odd}}{2m+1}$$

even = odd

Contradiction.
Therefore $x+y$ is even.
Q. E. D.

10. Consider the following theorem: "*If x is an odd integer, then $x + 2$ is odd.*"

a. Give a direct proof of this theorem.
b. Give a proof by contraposition.
c. Give a proof by contradiction.

a) Let $x = 2k+1$.
Then $x+2 = 2k+1+2 = 2(k+1)+1$,
which is odd. Q.E.D.

b) Suppose $x+2 = 2k$. Then $x = 2k-2$
$= 2(k-1)$,
which is even. Q.E.D.

c) Suppose x is odd but $x+2$ is even.

Then $x = 2k+1$ and $x+2 = 2\ell$.

Hence $(2k+1)+2 = 2\ell$.

Therefore $\underbrace{2(k+1-\ell)}_{\text{even}} = \underbrace{-1}_{\text{odd}}$.

Contradiction.

Q.E.D.

biconditional

$\leftrightarrow$

11. Prove that the following is true for all positive integers n: n is even if and only if $3n^2 + 8$ is even.

$\rightarrow$ If n is even, then $3n^2+8$ is even.
Direct Proof:

If n is even, then $n = 2k$ for some int. k.

Therefore $3n^2+8 = 3(2k)^2+8 = 12k^2+8$
$= 2(6k^2+4)$,

which is even.

If $3n^2+8$ is even, then n is even.

$\leftarrow$ Proof by contraposition:

Need to prove:

If n is odd, then $n = 2k+1$ for some k.

Therefore $3n^2+8 = 3(2k+1)^2+8 = 12k^2+12k+11$
$= 2(6k^2+6k+5)+1$, which is odd.

12. Prove that the equation $2x^2 + y^2 = 14$ has no positive integer solutions.

Q.E.D

Proof by cases.

There are only six cases that need to be considered:

$x=y=1$ $\quad$ $x=2, y=1$

$x=1, y=2$ $\quad$ $x=y=2$

$x=1, y=3$ $\quad$ $x=2, y=3$.

None of the cases is a solution.

Q.E.D.

13. On the island of knights and knaves you encounter two people, A and B. Person A says "B is a knave." Person B says "At least one of us is a knight." Determine whether each person is a knight or a knave.

~~knight~~	~~knight~~
~~knight~~	~~knave~~
knave	knight
~~knave~~	~~knave~~

15. Suppose $A = \{x, y\}$ and $B = \{x, \{x\}\}$. Mark the statement TRUE or FALSE.

a. $x \subseteq B$ False
b. $\emptyset \in \mathcal{P}(B)$ True
c. $\{x\} \subseteq A - B$ False
d. $|\mathcal{P}(A)| = 4$ True

$A - B = \{y\}$

$|\mathcal{P}(A)| = |\{\emptyset, \{x\}, \{y\}, \{x, y\}\}| = 4$

OR $|A| = 2$ $|\mathcal{P}(A)| = 2^2 = 4$

16. Determine if the given set is a power set of some set.

a. $\{\emptyset, \{a\}\}$
b. $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

a) YES Power set of $\{a\}$.

b) NO $|\{\emptyset, \{a\}, \{\emptyset, a\}\}| = 3$, impossible since not a power of 2.

17. Find the cardinality of the following sets.

a. $\mathcal{P}(A)$, where $A = \mathcal{P}(\{1, 2\})$
b. $\{x \mid x \in \mathbb{N} \text{ and } 4x^2 - 8 = 0\}$

a) $|A| = |\mathcal{P}(\{1,2\})| = 2^2 = 4$

$|\mathcal{P}(A)| = 2^4 = 16$

b) $4x^2 - 8 = 0$

$x^2 = 2$ No natural number solutions!

$|\{x \mid x \in \mathbb{N} \text{ and } 4x^2 - 8 = 0\}| = 0$

18.

Determine whether the set is finite or infinite. If the set is finite, find its size.

b. $\{1, 3, 5, 7, \ldots\}$.
c. $\mathcal{P}(A)$, where $A = \mathcal{P}(\{1, 2\})$.

b. Infinite

c. $|A| = 2^2 = 4$ $|\mathcal{P}(A)| = 2^4 = 16$