**Math 3336 Test 2 Review Questions**

**What is covered:** 2.3, 2.4, Chapter 4 (Sections 1-4), 5.1, 5.2.
**How to study:** Study the class notes, review homework problems, and try to do as many exercises as you can from the textbook. Note that answers are provided at the back of the book to all odd numbered problems.
Here I provide some examples for you. This is not a complete list, studying only these examples is not enough!

1. Use the Euclidean Algorithm to find $gcd(580, 50)$.

2. Express $gcd(84, 18)$ as a linear combination of 18 and 84.

3. Find an inverse of 5 modulo 12.

4. Solve the linear congruence $5x \equiv 3\ (mod\ 11)$.

5. Use Fermat's little theorem to find $9^{45}\ mod\ 23$.

6. Find the integer $a$ such that

    a. $a \equiv -15(mod\ 27)$ and $-70 \leq a \leq -50$
    b. $a \equiv 99(mod\ 41)$ and $100 \leq a \leq 140$

7. Decide whether each of these integers is congruent to 5 modulo 17.
    a. 56 b. 12 c. -29

8. Which positive integers less than 12 are relatively prime to 12?

9. Use the Principle of Mathematical Induction to prove that for all positive integers
$$1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6.$$

10. Prove that for every positive integer $n$,
$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3$$

11. Use the Principle of Mathematical Induction to prove that $n^3 > n^2 + 3$ for all $n \geq 2$.

12. Prove that $2^n > n^2$ if n is an integer greater than 4.

13. Use the Principle of Mathematical Induction to prove that $3|(n^3 + 3n^2 + 2n)$ for all $n \geq 1$.

14. Prove that $n^2 - 1$ is divisible by 8 whenever $n$ is an odd positive integer.

15. Use the Principle of Mathematical Induction to prove that any integer amount of postage from 18 cents on up can be made from an infinite supply of 4-cent and 7-cent stamps.

    Do you use the Principle of Mathematical Induction or the Strong Principle of Mathematical Induction to prove this result?

16. A sequence $a_1, a_2, a_3, \ldots$ is defined recursively by $a_1 = 1, a_2 = 4$ and $a_n = 2a_{n-1} - a_{n-2} + 2$ for $n \geq 3$.
    Conjecture a closed formula for $a_n$ and prove the formula using the Principle of Mathematical Induction.

    Do you use the Principle of Mathematical Induction or the Strong Principle of Mathematical Induction to prove this result?

17. Suppose $f: \mathbb{Z} \to \mathbb{Z}$ has the rule $f(n) = 3n^2 - 1$. Determine whether $f$ is 1-1.

18. Suppose $f: \mathbb{Z} \to \mathbb{Z}$ has the rule $f(n) = 3n^2 - 1$. Determine whether f is onto $\mathbb{Z}$

19. Suppose $g: A \to B$ and $f: B \to C$ where $A = \{1,2,3,4\}, B = \{a,b,c\}, C = \{2,7,10\}$, and $f$ and $g$ are defined by $g = \{(1,b),(2,a),(3,a),(4,b)\}$ and $f = \{(a,10),(b,7),(c,2)\}$. Find $f \circ g$.

20. Suppose $g: A \to B$ and $f: B \to C$ where $A = \{1,2,3,4\}, B = \{a,b,c\}, C = \{2,7,10\}$, and $f$ and $g$ are defined by $g = \{(1,b),(2,a),(3,a),(4,b)\}$ and $f = \{(a,10),(b,7),(c,2)\}$. Find $f^{-1}$.

21. Find a formula that generates the following sequence $\{1, 1/3, 1/5, 1/7, 1/9, \ldots\}$.

22. Describe the following sequence recursively. Include the initial condition and assume the sequence begins with $a_1$. $\{1/2, 1/3, 1/4, 1/5, \ldots\}$.

# Math 3336 Test 2 Review Questions Solutions

1. $\gcd(580, 50) = \boxed{10}$

$580 = 11 \cdot 50 + 30$
$50 = 1 \cdot 30 + 20$
$30 = 1 \cdot 20 + 10$
$20 = 2 \cdot 10 + 0$

2. $\gcd(84, 18) = 6$

$84 = 4 \cdot 18 + 12 \rightarrow 12 = 84 - 4 \cdot 18$
$18 = 1 \cdot 12 + 6 \rightarrow 6 = 18 - 1 \cdot 12$
$12 = 2 \cdot 6 + 0$

$$= 18 - 1(84 - 4 \cdot 18)$$
$$= 18 - 1 \cdot 84 + 4 \cdot 18$$
$$= 5 \cdot 18 - 1 \cdot 84$$

$$\boxed{6 = 5 \cdot 18 - 1 \cdot 84}$$

3. Inverse of 5 mod 12

$12 = 2 \cdot 5 + 2 \rightarrow 2 = 12 - 2 \cdot 5$
$5 = 2 \cdot 2 + 1 \rightarrow 1 = 5 - 2 \cdot 2$
$2 = 2 \cdot 1 + 0$

$$= 5 - 2(12 - 2 \cdot 5)$$
$$= 5 - 2 \cdot 12 + 4 \cdot 5$$
$$= 5 \cdot 5 - 2 \cdot 12$$

$1 = 5 \cdot 5 - 2 \cdot 12$

$$\boxed{5 \cdot 5 \equiv 1 \pmod{12}}$$

4. $5x \equiv 3 \pmod{11}$

$11 = 2\cdot 5 + 1 \rightarrow 1 = 11 - 2\cdot 5$

$5 = 5\cdot 1 + 0$

$-2$ is inverse of 5 mod 11

$(-2)5x \equiv 3(-2) \mod 11$

$x \equiv -6 \mod 11$

$x \equiv 5 \mod 11$

$\boxed{x = 5 + 11k}$

5. $9^{45} \mod 23$ $\quad a^{p-1} \equiv 1 \pmod{p}$

$p = 23$

$9^{22} \equiv 1 \pmod{23}$

$9^{45} \equiv (9^{22})^2 \cdot 9 \equiv (1)^2 \cdot 9 \equiv \boxed{9 \mod 23}$

6. a) $a \equiv -15 \pmod{27}$ and $-70 \le a \le -50$

$\boxed{a = -69}$ $(-15 - 27 - 27 = -69)$

b) $a \equiv 99 \pmod{41}$ and $100 \le a \le 140$

$\boxed{a = 140}$ $(99 + 41 = 140)$

7. a) $56 \overset{?}{\equiv} 5 \pmod{17}$ Yes

$17 \overset{?}{\mid} (56 - 5)$ Yes

b) $12 \overset{?}{\equiv} 5 \pmod{17}$ No

$17 \overset{?}{\mid} (12 - 5)$ No

c) $-29 \overset{?}{\equiv} 5 \pmod{17}$ Yes

$17 \overset{?}{\mid} (-29 - 5)$ Yes

8. Positive integers less than 12:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11

Relatively prime to 12: 1, 5, 7, 11

9.

Basis step: P(1)

$$1^2 = \frac{1(1+1)(2(1)+1)}{6}$$

$$1 = 1 \checkmark$$

Inductive Step: $P(k) \rightarrow P(k+1)$

Assume $1^2+2^2+\ldots+k^2 = k(k+1)(2k+1)/6$, then

show $1^2+2^2+\ldots+k^2+(k+1)^2 = (k+1)(k+2)(2k+3)/6$

$$\underbrace{1^2+2^2+\ldots+k^2}_{\text{Ind. Hyp.}}+(k+1)^2 = \frac{k(k+1)(2k+1)}{6}+\frac{6(k+1)^2}{6}$$

$$= \frac{(k+1)}{6}\left[k(2k+1)+6(k+1)\right]$$

$$= \frac{(k+1)}{6}(2k^2+k+6k+6) = \frac{(k+1)}{6}(2k^2+7k+6) =$$

Note: $2k^2+7k+6 = 2k^2+4k+3k+6$
$= 2k(k+2)+3(k+2)$
$= (k+2)(2k+3)$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Both the basis and the inductive steps are complete. So by the principle of math. induction the statement is true for every positive integer n. ∎

10. Basis step: P(1)

$$1\cdot 2 = \frac{\overset{1\cdot 2\cdot 3}{1(1+1)(1+2)}}{3}$$

$$2 = 2 \checkmark$$

Inductive step: $P(k) \Rightarrow P(k+1)$

Assume $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + k(k+1) = \frac{k(k+1)(k+2)}{3}$,

then show $1\cdot 2 + 2\cdot 3 + \ldots + k(k+1) + (k+1)(k+2) =$

$$= \frac{(k+1)(k+2)(k+3)}{3}.$$

$$\underbrace{1\cdot 2 + 2\cdot 3 + \ldots + k(k+1)}_{\text{I. H.}} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3}$$

$$= \frac{(k+1)(k+2)}{3}\left[k+3\right] = \frac{(k+1)(k+2)(k+3)}{3}.$$

Both the basis and the inductive steps are complete. So by the principle of math. induction the statement is true for every positive integer $n$.

$\blacksquare$.

11. $n^3 > n^2+3$ for all $n \geq 2$

Basis step: $(2)^3 > (2)^2+3$
$P(2)$ $8 > 7$ ✓

Inductive step: Assume: $k^3 > k^2+3$, then
$P(k) \Rightarrow P(k+1)$ show $(k+1)^3 > (k+1)^2+3$

$$(k+1)^2+3 = k^2+2k+1+3 = (k^2+3)+2k+1 \underset{IH}{<}$$
$$< k^3+2k+1 \leq k^3+3k+1$$
$$\leq k^3+3k^2+3k+1$$
$$= (k+1)^3 \checkmark$$

Both the basis and the inductive steps are complete. So by the principle of math. induction the statement is true for every $n \geq 2$. ∎

12. Basis step: $P(5)$

$$2^5 > 5^2$$
$$32 > 25 \checkmark$$

Inductive step: $P(k) \rightarrow P(k+1)$

Assume $2^k > k^2$, then show $2^{k+1} > (k+1)^2$

$$2^{k+1} = 2 \cdot 2^k = 2^k + 2^k \underset{\text{I.H.}}{>} k^2 + k^2$$
$$= k^2 + k \cdot k$$
$$> k^2 + 4k \quad \text{since } k > 4$$
$$\geq k^2 + 2k + 1$$
$$= (k+1)^2.$$

Both the basis and the inductive steps are complete. Hence by the principle of math. induction the statement is true for every $n > 4$. $\blacksquare$.

13. $3 \mid (n^3+3n^2+2n)$ for all $n \geq 1$

Basic step: $P(1)$ $3 \mid 1^3+3(1)^2+2(1)$ $3 \mid 6$ ✓

Inductive step: Assume $3 \mid k^3+3k^2+2k$, then

$P(k) \Rightarrow P(k+1)$ show $3 \mid (k+1)^3+3(k+1)^2+2(k+1)$

$$(k+1)^3+3(k+1)^2+2(k+1)$$

$$= k^3+3k^2+3k+1+3(k^2+2k+1)+2k+2$$

$$= \underline{k^3+3k^2}+3k+3+\underline{2k}+3k^2+6k+3$$

$$= \underbrace{(k^3+3k^2+2k)}_{IH}+3(k^2+3k+2)$$

$$3 \mid (k^3+3k^2+2k)+3(k^2+3k+2)$$

3 divides each term. ∎

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14. Prove that $n^2-1$ is divisible by 8 whenever $n$ is an odd positive integer.

$n = 2m-1$

Then the statement is $(2m-1)^2-1$ is divisible by 8 for all positive integers $m$.

∎

Basis step: P(1) $\frac{0}{8}=0$ so $8|0$. ✓

Inductive step: $P(k) \to P(k+1)$

Assume $8|(2k-1)^2-1$, then show

that $8|(2(k+1)-1)^2-1$

$$(2(k+1)-1)^2-1$$
$$=(2k+2-1)^2-1 = ((2k-1)+2)^2-1$$
$$=(2k-1)^2+4(2k-1)+4-1$$
$$=(2k-1)^2-1+8k\cancel{-4}\cancel{+4}$$
$$=\underbrace{(2k-1)^2-1}+8k$$

$8|(2k-1)^2-1$ by I.H.

$8|8k$

Then $8|(2k-1)^2-1+8k$. ∎

15. Any integer amount of postage from 18 cents on up can be made from an infinite supply of 4-cent and 7-cent stamps.

Basis step: Use two 7-cent stamps and
P(18) one 4-cent stamp.

Inductive hypothesis: Assume can make
$P(k) \to P(k+1)$ a postage worth $k$ cents,
then show that can make
a postage worth $k+1$ cents.

Case 1: A $k$-cent postage has a 7-cent stamp.
Replace one 7-cent stamp with two
4-cent stamps.

Case 2: A $k$-cent postage does <u>not</u> use
7-cent stamps only 4-cent stamps.
Then there must be at least five
4-cent stamps.
Replace five 4-cent stamps with
three 7-cent stamps. ✓

Both the basis and the inductive steps are completed.
Therefore by the Strong Principle of Math Induction the proof of the statement is complete. ■

16. $a_1 = 1$ $\quad a_n = 2a_{n-1} - a_{n-2} + 2$ for $n \geq 3$

$a_2 = 4$

$a_3 = 2a_2 - a_1 + 2 = 2 \cdot 4 - 1 + 2 = 9$

$a_4 = 2a_3 - a_2 + 2 = 2 \cdot 9 - 4 + 2 = 16$

$a_5 = 2a_4 - a_3 + 2 = 2 \cdot 16 - 9 + 2 = 25$

Conjecture: $a_n = n^2$ for $n \geq 1$

The Strong Principle of Math. Induction.

Basis step: $a_1 = 1 = 1^2$ ✓

Inductive step: Assume $a_i = i^2$ for all $i$ with $1 \leq i \leq k$, then show $a_{k+1} = (k+1)^2$

$$\begin{aligned}
a_{k+1} &= 2a_k - a_{k-1} + 2 \\
&= 2k^2 - (k-1)^2 + 2 = 2k^2 - (k^2 - 2k + 1) + 2 \\
&= 2k^2 - k^2 + 2k - 1 + 2 \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

We completed both the basis and the inductive steps. Therefore by the Strong Principle of Math. Induction $a_{k+1} = (k+1)^2$ ■.

17. Suppose $f: \mathbb{Z} \to \mathbb{Z}$ has the rule $f(n) = 3n^2 - 1$. Determine whether $f$ is 1-1.

$\boxed{\text{NO}}$

$f(1) = 3(1)^2 - 1 = 2$

$f(-1) = 3(-1)^2 - 1 = 2$

18. Suppose $f: \mathbb{Z} \to \mathbb{Z}$ has the rule $f(n) = 3n^2 - 1$. Determine whether f is onto $\mathbb{Z}$.

$\boxed{\text{NO}}$

$3n^2 - 1 \neq -2$

$3n^2 \neq -1$

$n^2 \neq -1/3$

19. Suppose $g: A \to B$ and $f: B \to C$ where $A = \{1,2,3,4\}$, $B = \{a,b,c\}$, $C = \{2,7,10\}$, and $f$ and $g$ are defined by $g = \{(1,b),(2,a),(3,a),(4,b)\}$ and $f = \{(a,10),(b,7),(c,2)\}$. Find $f \circ g$.

$(f \circ g)(x) = f(g(x))$

$f(g(1)) = f(b) = 7$

$f(g(2)) = f(a) = 10$

$f(g(3)) = f(a) = 10$

$f(g(4)) = f(b) = 7$

$\{(1,7),(2,10),(3,10),(4,7)\}$

20. Suppose $g: A \to B$ and $f: B \to C$ where $A = \{1,2,3,4\}$, $B = \{a,b,c\}$, $C = \{2,7,10\}$, and $f$ and $g$ are defined by $g = \{(1,b),(2,a),(3,a),(4,b)\}$ and $f = \{(a,10),(b,7),(c,2)\}$. Find $f^{-1}$.

$a \overset{f^{-1}}{\underset{f}{\rightleftarrows}} 10$

$b \overset{f^{-1}}{\underset{f}{\rightleftarrows}} 7$

$c \overset{f^{-1}}{\underset{f}{\rightleftarrows}} 2$

$\{(2,c),(7,b),(10,a)\}$

21. Find a formula that generates the following sequence $\{1, 1/3, 1/5, 1/7, 1/9, \ldots\}$.

$$a_n = \frac{1}{2n-1}$$

22. Describe the following sequence recursively. Include the initial condition and assume the sequence begins with $a_1$. $\{1/2, 1/3, 1/4, 1/5, \ldots\}$.

$$a_n = \frac{a_{n-1}}{1+a_{n-1}}$$