Math 3336 Test 3 Review Questions

How to study: Study the class notes, review homework problems, and try to do as many exercises as you can from the textbook. Note that answers are provided at the back of the book to all odd numbered problems.
Here I provide some examples for you. This is not a complete list, studying only these examples is not enough!

1. Describe the following sequence recursively. Include initial conditions and assume that the sequences begin with a_1.
   a_n = the number of bit strings of length n with an even number of 0's.

2. Determine whether the recurrence relation is a linear homogeneous recurrence relation with constant coefficients.
   a. $a_n = 0.7a_{n-1} - 0.3a_{n-2}$
   b. $a_n = na_{n-1}$
   c. $a_n - 3a_{n-1} + 4a_{n-2} = 0$

3. Solve the recurrence relation either by using the characteristic equation or by discovering a pattern formed by the terms.
   a. $a_n = 5a_{n-1} - 4a_{n-2},\ a_0 = 1, a_1 = 0$
   b. $a_n = 3a_{n-1},\ a_0 = 2$
   c. $a_n = 2a_{n-1} + 2a_{n-2},\ a_0 = 0, a_1 = 1$
   d. $a_n = 2a_{n-1} + 5,\ a_0 = 3$

4. Assume that the characteristic equation for a homogeneous linear recurrence relation with constant coefficients is $(r - 5)^3 = 0$. Describe the form for the general solution to the recurrence relation.

5. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + F(n)$ have when $F(n) = n2^n$?

6. Find $|A_1 \cup A_2 \cup A_3 \cup A_4|$ if each set A_i has 150 elements, each intersection of two sets has 80 elements, each intersection of three sets has 20 elements, and there are no elements in all four sets.

7. Find the number of positive integers ≤ 1000 that are multiples of at least one of 2, 6, 12.

8. There are 2504 computer science students at a school. Of these, 1876 have taken a course in Java, 999 have taken a course in Linux, and 345 have taken a course in C. Further, 876 have taken courses in both Java and Linux, 231 have taken courses in both Linux and C, and 290 have taken courses in both Java and C. If 189 of these students have taken courses in Linux, Java, and C, how many of these 2504 students have not taken a course in any of these programming languages?

9. How many 4 digit numbers can we make using digits 2, 3, 4, and 5, if we are allowed to repeat digits and must use 5 at least once.

10. How many positive integers less than 1000
    a. have exactly three decimal digits?
    b. have an odd number decimal digits?
    c. have at least one decimal digit equal to 9?
    d. have no odd decimal digits?
    e. have two consecutive decimal digits equal to 5?
    f. are palindromes (that is read same forward and backward)?

11. Nine people (Ann, Ben, Cal, Dot, Ed, Fran, Gail, Hal, and Ida) are in a room. Five of them stand in a row for a picture.
    a. In how many ways can this be done if Ben is to be in the picture?
    b. In how many ways can this be done if Dot is on the left end and Ed is on the right end?
    c. In how many ways can this be done if Ed and Gail are in the picture, standing next to each other?

12. Suppose you have 30 books (15 novels, 10 history books, and 5 math books). Assume that all 30 books are different.
    a. In how many ways can you put the 30 books in a row on a shelf?
    b. In how many ways can you get a bunch of four books to give to a friend?
    c. In how many ways can you get a bunch of three history books and seven novels to give to a friend?
    d. In how many ways can you put the 30 books in a row on a shelf if the novels are on the left, the math books are in the middle, and the history books are on the right?

13. Find the number of subsets of $S = \{1, 2, 3, \ldots, 10\}$ that contain both 5 and 6.

14. Show that if five points are picked on or in the interior of a square of side length 2, then there are at least two of these points no farther than $\sqrt{2}$ apart.

15. A professor teaching a Discrete Math course gives a multiple choice quiz that has ten questions, each with four possible responses: a, b, c, d. What is the minimum number of students that must be in the professor's class in order to guarantee that at least three answer sheets must be identical? (Assume that no answers are left blank.)

16. A factory makes automobile parts. Each part has a code consisting of a digit, a letter, and a digit, with the digits distinct, such as 5C7, 1O6, or 3Z0. Last week the factory made 5,000 parts. Find the minimum number of parts that must have the same serial number.

17.
    a. In how many ways are there to arrange the letters of the word NONSENSE?
    b. How many of these ways start or end with the letter O?

18. You have 50 of each of the following kinds of jellybeans: red, orange, green, yellow. The jellybeans of each color are identical.
    a. In how many ways can you put all the jellybeans in a row?
    b. How many handfuls of 12 are possible?

19. Find the number of solutions to $x + y + z = 32$, where x, y, and z are nonnegative integers.

20. 15 club members donated \$100 to the club fund. If every club member donated an integer number of dollars, prove that there are two club members who donated the same amount.

Math 3336 Test 3 Review KEY

1. a_n = number of bit strings of length n with an even number of zeroes.

$a_1 = 1$ string = 1 0 zeroes

We may form a string of length n with an even number of zeroes in two ways.

1. From a string of length $n-1$ with an even number of zeroes by adding 1 at the end, i.e. in

$$a_{n-1} \text{ ways}$$

2. From an invalid (with an odd # of zeroes) string of length $n-1$ by adding a 0 at the end.

There are $2^{n-1} - a_{n-1}$ invalid strings.

So, the total # $a_n = a_{n-1} + 2^{n-1} - a_{n-1} = 2^{n-1}$

$a_n = 2^{n-1}$ is NOT recursively defined

since $a_{n-1} = 2^{n-2}$, then

$$a_n = 2 \cdot 2^{n-2} = 2a_{n-1}. \quad \boxed{a_n = 2a_{n-1}}$$

2. a) YES b) NO c) YES

3. a) $a_n = 5a_{n-1} - 4a_{n-2}$ $a_0 = 1$ $a_1 = 0$

$c_1 = 5$ $c_2 = -4$ $k = 2$

$r^2 - 5r + 4 = 0$
$(r-4)(r-1) = 0$
$r = 4$ $r = 1$

$a_n = \alpha_1 \cdot 4^n + \alpha_2 (1)^n$

$a_0 = 1 = \alpha_1 \cdot \underbrace{\cancel{4^0}}_{1} + \alpha_2 \underbrace{\cancel{(1)^0}}_{1}$

$a_1 = 0 = \alpha_1 \cdot 4^1 + \alpha_2 (1)^1$

$-\alpha_1 - \alpha_2 = -1$
$+\;\; 4\alpha_1 + \alpha_2 = 0$

$3\alpha_1 = -1$

$\alpha_1 = -1/3$

$\alpha_2 = 4/3$

$$\boxed{a_n = (-1/3) \cdot 4^n + (4/3)(1)^n}$$

b) $a_n = 3a_{n-1} \quad a_0 = 2$

$$a_n = 3(a_{n-1}) = 3(3a_{n-2}) = 3^2 a_{n-2}$$

$$= 3^2(3a_{n-3}) \text{ etc.}$$

$$\boxed{a_n = 2 \cdot 3^n}$$

c)

$$a_n = 2a_{n-1} + 2a_{n-2} \qquad a_0 = 0 \quad a_1 = 1$$

$$c_1 = 2 \quad c_2 = 2 \quad k = 2$$

$$r^2 - 2r - 2 = 0 \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(-2)}}{2}$$

$$r_{1,2} = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2}$$

$$\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$$

$$r_1 = 1 + \sqrt{3} \quad r_2 = 1 - \sqrt{3}$$

$$a_n = \alpha_1 (1+\sqrt{3})^n + \alpha_2 (1-\sqrt{3})^n$$

$$a_0 = 0 = \alpha_1 + \alpha_2$$

$$a_1 = 1 = \alpha_1 (1+\sqrt{3}) + \alpha_2 (1-\sqrt{3})$$

$\alpha_1 + \alpha_2 = 0 \qquad \alpha_2 = -\alpha_1$

$\alpha_1(1+\sqrt{3}) + \alpha_2(1-\sqrt{3}) = 1$

$\alpha_1(1+\sqrt{3}) - \alpha_1(1-\sqrt{3}) = 1$

$\cancel{\alpha_1} + \sqrt{3}\alpha_1 - \cancel{\alpha_1} + \sqrt{3}\alpha_1 = 1$

$2\sqrt{3}\,\alpha_1 = 1 \qquad \alpha_1 = \frac{1 \cdot \sqrt{3}}{2\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{3}}{6}$

$\alpha_1 = \frac{\sqrt{3}}{6} \qquad \alpha_2 = -\frac{\sqrt{3}}{6}$

$$\boxed{a_n = \frac{\sqrt{3}}{6}(1+\sqrt{3})^n - \frac{\sqrt{3}}{6}(1-\sqrt{3})^n}$$

d) $a_n = 2a_{n-1} + 5 \qquad a_0 = 3$

$$\begin{aligned} a_n &= 2(2a_{n-2}+5)+5 \\ &= 2^2 a_{n-2} + 2\cdot 5 + 5 \\ &= 2^2(2a_{n-3}+5) + 2\cdot 5 + 5 \\ &= 2^3 a_{n-3} + 2^2\cdot 5 + 2\cdot 5 + 5 \end{aligned}$$ etc.

$$\boxed{a_n = 3\cdot 2^n + 5(2^n - 1)}$$ geom. prog.

4.

$$(r-5)^3 = 0$$

$$r = 5 \quad \text{mult.} = 3$$

$$a_n = \alpha_1 5^n + \alpha_2 n 5^n + \alpha_3 n^2 5^n$$

5.

$$n^2(p_1 n + p_0) 2^n$$ See Thm 6 p 523

6.

$$|A_1 \cup A_2 \cup A_3 \cup A_4|$$

$$= \underbrace{|A_1|}_{150} + \underbrace{|A_2|}_{150} + \underbrace{|A_3|}_{150} + \underbrace{|A_4|}_{150}$$

$$- \underbrace{|A_1 \cap A_2|}_{80} - \underbrace{|A_2 \cap A_3|}_{80} - \underbrace{|A_1 \cap A_4|}_{80} - \underbrace{|A_2 \cap A_3|}_{80}$$

$$- \underbrace{|A_2 \cap A_4|}_{80} - \underbrace{|A_3 \cap A_4|}_{80}$$

$$+ \underbrace{|A_1 \cap A_2 \cap A_3|}_{20} + \underbrace{|A_1 \cap A_2 \cap A_4|}_{20} + \underbrace{|A_1 \cap A_3 \cap A_4|}_{20}$$

$$+ \underbrace{|A_2 \cap A_3 \cap A_4|}_{20} - \underbrace{|A_1 \cap A_2 \cap A_3 \cap A_4|}_{0}$$

$$= 4(150) - 6(80) + 4(20) - 0 = \boxed{200}$$

7. $A =$ multiples of 2 $\quad |A| = \left\lfloor \frac{1000}{2} \right\rfloor = 500$

$B =$ multiples of 6 $\quad |B| = \left\lfloor \frac{1000}{6} \right\rfloor = 166$

$C =$ multiples of 12 $\quad |C| = \left\lfloor \frac{1000}{12} \right\rfloor = 83$

$|A \cap B| = \left\lfloor \frac{1000}{6} \right\rfloor = 166$

Anything divisible by 6 is divisible by 2.

$|A \cap C| = \left\lfloor \frac{1000}{12} \right\rfloor = 83$

Anything divisible by 12 is divisible by 2.

$|B \cap C| = \left\lfloor \frac{1000}{12} \right\rfloor = 83$

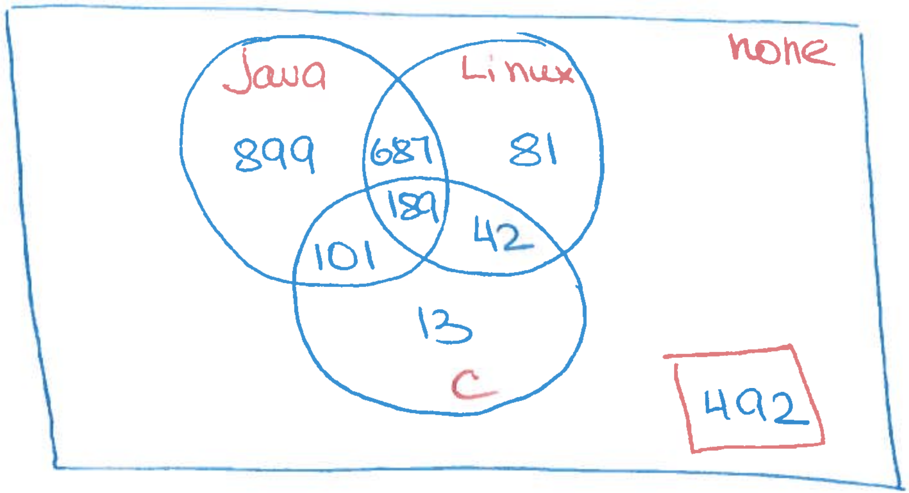
Anything divisible by 12 is divisible by 6.

$|A \cap B \cap C| = \left\lfloor \frac{1000}{12} \right\rfloor = 83$

$$|A \cup B \cup C| = \overset{500}{|A|} + \overset{166}{|B|} + \overset{83}{|C|} - \overset{166}{|A \cap B|} - \overset{83}{|A \cap C|}$$
$$- \overset{83}{|B \cap C|} + \overset{83}{|A \cap B \cap C|}$$
$$= \boxed{500}$$

Also see Example 2 in the textbook (page 553)

8.



Also see Examples 3 and 4 (page ~~554~~ - 555) in the textbook!

9. $4^4 - 3^4 = 175$

Count all 4 digit numbers we can make using 2, 3, 4, 5, then subtract those that do not have 5 in them.

10. a) Integers from 100 − 999 have 3 decimal digits.

$999 - 100 + 1 = \boxed{900}$

b) + 9 one-digit integers

$900 + 9 = \boxed{909}$

c) $10^3 - 9^3 = \boxed{271}$

d) $5^3 = 125$ 0 1 2 3 4 5 6 7 8 9

0 is not possible

$125 - 1 = \boxed{124}$

e) $d55 \quad d \neq 5$

$55d$

$555 \qquad 9+9+1 = \boxed{19}$

f) one-digit #s $\rightarrow 9$

two-digit #s $\rightarrow 9$

$\underset{\substack{9 \\ \text{choices}}}{\overset{\uparrow}{\_\_}} \quad \underset{\text{no choice}}{\overset{\uparrow}{\_\_}}$

three-digit #s

$\underset{\substack{9 \\ \text{choices}}}{\overset{\uparrow}{\_\_}} \quad \underset{10}{\overset{\uparrow}{\_\_}} \quad \underset{\substack{\text{no} \\ \text{choices}}}{\overset{\uparrow}{\_\_}} \rightarrow 90$

$9+9+90 = \boxed{108}$

11.

a) $5P(8,4)$

b) $P(7,3)$

c) $2 \cdot 4 \cdot P(7,3)$

12.

30 Books

- 15 Novels
- 10 History Books
- 5 Math Books

a) $30!$

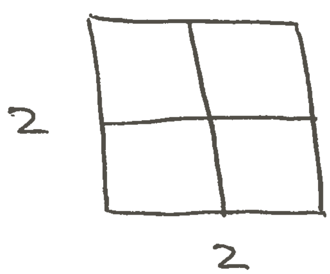
b) $C(30,4)$

c) $C(10,3) \cdot C(15,7)$

d) $15! \cdot 5! \cdot 10!$

13. 2^8

14.



Divide a square into 4 1×1 inch squares.
By the Pigeon hole Principle there are 2 points in one of these squares.
The largest distance between any two points in a 1×1 inch square is $\sqrt{2}$ (diagonal).

15. 10 questions
4 answer choices for each question

There are 4^{10} possibe answer sheets.

$2 \cdot 4^{10} + 1$ is the minimum number that will guarantee three identical answer sheets.

16. The # of codes is $10 \cdot 26 \cdot 9 = 2{,}340$

$\left\lceil \frac{5{,}000}{2340} \right\rceil = 3$. Hence at least 3 parts have the same code number.

17. a) NONSENSE
3 - Ns
2 - Ss
2 - Es

$$\frac{8!}{3!\,2!\,2!}$$

b) $$\frac{2 \cdot 7!}{3!\,2!\,2!}$$

18. red, orange, green, yellow

a) $\frac{200!}{(50!)^4}$

b) $C(15,3)$

19. $C(34,2)$

20. If all 15 club members donate different amounts, then the total donation is

$1+2+3+4+5...+15=105$

Contradiction, since the total donation is \$100.

There must be two club members that donated the same amount.