Section 4.1: Exponential Growth and Decay

A function that grows or decays by a constant percentage change over each fixed change in input is called an exponential function.

Exponents – A quick review

1. \( a^0 = 1 \)
2. \( a^{-n} = \frac{1}{a^n} \)
3. \( a^n = \sqrt[n]{a} \)
4. \( \frac{a^m}{a^n} = a^{m-n}, a \neq 0 \)
5. \( (a^m)^n = a^{mn} \)
6. \( (ab)^n = a^n b^n \)
7. \( (a^m)^n = a^{mn} \)
8. \( (a^m)^n = a^{mn} \)
9. \( \frac{a^m}{b^n} = a^{m-n}, b \neq 0 \)
10. For \( b \neq 1 \), \( b^x = b^y \) means \( x = y \).

Example 1: Simplify. Write your final answer without negative exponents.

1. \( ((a^2)^3)^4 \)

\[ (a^6)^4 \]

\[ \Rightarrow a^{24} \]

2. \( \frac{a^3}{a^2 b^3} \)

\[ a^{3-2} \cdot \frac{1}{b^{2-3}} \]

\[ a \cdot b^{-1} \Rightarrow \frac{a}{b} \]

3. \( a^3 b^2 a^4 b^{-1} \)

\[ a^3 a^4 \cdot b^2 \cdot b^{-1} \]

\[ \Rightarrow a^7 b \]

4. \( \frac{a^{-4} b^{-3}}{a^{-6} b^{-2}} \)

\[ a^{-4} \cdot b^{-3} \cdot (a^2) \]

\[ a^2 \cdot b^{-1} = a^2 \cdot \frac{1}{b} \]

\[ \Rightarrow a^2 b \]
Exponential Growth

Example 2: A petri dish contains 500 bacteria at the start of an experiment. The number of bacteria double each hour. We can calculate the number of bacteria in the dish as a function of the number of hours since the experiment started. Here is the beginning of the chart:

<table>
<thead>
<tr>
<th>Time, in hours</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of bacteria</td>
<td>500</td>
<td>1000</td>
<td>2000</td>
<td>4000</td>
<td>8000</td>
<td>16,000</td>
</tr>
</tbody>
</table>

Let $N = N(t)$ be the number of bacteria in the dish $t$ hours after the experiment started.

Let’s use the pattern in the chart above to develop a formula for $N(t)$.

\[
N(0) = 500 \\
N(1) = 2 \cdot N(0) = 2(500) = 1000 \\
N(2) = 2 \cdot N(1) = 2(1000) = 2000 \\
N(3) = 2 \cdot N(2) = 2(2000) = 4000 \\
N(4) = 2 \cdot N(3) = 2(4000) = 8000 \\
N(5) = 2 \cdot N(4) = 2(8000) = 16,000
\]

The growth factor is: $2$

\[
\Rightarrow N(t) = N(0) \cdot 2^t
\]
The Exponential Growth Formula: \( N(t) = Pa^t, a > 1. \)

\( P \) is the initial value, \( t \) is the time and \( a \) is the growth factor for each unit of time.

**Exponential Decay**

**Example 3:** Suppose we change the experiment in Example 2 by introducing an antibiotic into the petri dish. Now, the number of bacteria in the dish is cut in half each hour.

<table>
<thead>
<tr>
<th>Time, in hours</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of bacteria</td>
<td>500</td>
<td>250</td>
<td>125</td>
<td>62.5</td>
<td>31.25</td>
<td>15.625</td>
</tr>
</tbody>
</table>

Let \( N(t) \) be the number of bacteria in the petri dish \( t \) hours after the antibiotic was introduced. The pattern in the above chart suggests the following formula for \( N(t) \):

\[
N(0) = 500
\]

\[
N(1) = \frac{1}{2} N(0) = \frac{1}{2} (500) = 250
\]

\[
N(2) = \frac{1}{2} N(1) = \frac{1}{2} \left( \frac{1}{2} (500) \right) = \ldots = 125
\]

\[
N(3) = \frac{1}{2} N(2) = \frac{1}{2} \left( \frac{1}{2} (500) \right) = \frac{1}{2} \left( \frac{1}{2} \left( \frac{1}{2} (500) \right) \right)
\]

The decay factor is:

\[
\frac{1}{2}
\]

\[\Rightarrow \quad N(t) = N(0) \left( \frac{1}{2} \right)^t\]

The Exponential **Decay** Formula: \( N(t) = Pa^t, a < 1. \)

Q2: \( \sqrt{\text{for } N(t) = 500 (0.99)^t} \)

\[\text{a)} \quad \text{growth} \quad 0.99 < 1 \]

\[\text{b)} \quad \text{decay} \]
**Example 4: Radioactive Decay**

If there is 1 gram of heavy hydrogen in a container, then as a result of radioactive decay there will be 0.783 grams of heavy hydrogen in the container one year later. Suppose a container starts with 25 grams of hydrogen.

a. Find the formula for the number of grams of hydrogen in the container as a function of the time \( t \) in years.

b. How much heavy hydrogen is left after 5 years?

c. Plot the graph of the function.

d. Find the time \( t \) when \( \frac{1}{2} \) of the hydrogen is left in the container.

---

**Part a**

\[
H(t) = 25 \cdot (0.783)^t
\]

---

**Part b**

\[
H(5) = 25 \cdot (0.783)^5 = 7.358 \text{ grams}
\]

---

**Part c**

Graph showing the decay over time.

---

**Part d**

\[
25 \cdot (0.783)^t = 12.5 \quad \Rightarrow \quad (0.783)^t = \frac{1}{2}
\]

\[
\ln (0.783) = \ln \left( \frac{1}{2} \right)
\]

\[
\Rightarrow t = \frac{\ln \left( \frac{1}{2} \right)}{\ln (0.783)}
\]
**Constant Proportional Change**

A function is exponential if it shows constant percentage (or proportional) growth or decay.

**Growth:** For an exponential function with discrete (yearly, monthly, etc.) percentage growth rate \( r \) as a decimal, the growth factor \( a = 1 + r \).

**Decay:** For an exponential function with discrete (yearly, monthly, etc.) percentage decay rate \( r \) as a decimal, the decay factor \( a = 1 - r \).

**Example 5:** A certain phenomenon has an initial value of 23 and grows at a rate of 6% per year. Give an exponential function which describes this phenomenon.

\[
\alpha = 1 + r = 1 + 0.06 = 1.06
\]

\[
f(t) = 23 \left( 1 + 0.06 \right)^t
\]

**Example 6:** Suppose the amount of pollution in a tank starts at 100 pounds and decreases by 16% per hour. Find the decay constant and the formula for the amount of pollutant in the tank in pounds as a function of time in hours. How much is left in the tank after 10 hours? 15 hours?

\[
\beta = 1 - r = 1 - 0.16
\]

\[
P(t) = 100 \left( 1 - 0.16 \right)^t
\]

\[
P(10) = 100 \left( 0.84 \right)^{10} \approx 17.49 \text{ lbs}
\]

\[
P(15) = 100 \left( 0.84 \right)^{15} \approx 7.31 \text{ lbs}
\]
Growth or Decay Factor Unit Conversion
If the growth or decay factor for one period of time is \( a \), then the growth or decay factor for \( k \) periods of time is given by \( A = a^k \).

Here is the conversion diagram:

What does this mean in practical terms?

**Example 7:** Census data is collected every 10 years. Suppose the census data shows that the population increases by 23% per decade. What is the yearly growth factor?

\[
G = 0.23 \quad \text{in} \quad 10 \text{ years}
\]

\[
a = 1 + r = 1 + 0.23 = 1.23
\]

\[
10\text{th root}
\]

\[
(1.23)^{1/10}
\]

\[
G = 1.0209
\]

Every year, this is the overall growth

\[
1.0209 = a = 1 + r
\]

\[
r = 0.0209
\]

\[
2.09 \% \text{ growth per year}
\]
Example 8: Terry deposits $10,000 in an account with a 0.75% monthly interest rate. What is the yearly growth factor?

\[ a = 1 + r \]
\[ = 1 + 0.0075 \]
\[ = 1.0075 \]

\[ \text{GF per month} \]

\[ \left( a \right)^{12} \]
\[ \Rightarrow (1.0075)^{12} \]
\[ \Rightarrow \text{yearly IR} \]

Find the amount in the account after 10 years.

1. \[ 10000 \left( 1.0938 \right)^{10} \]
\[ \Rightarrow 24,513.57 \]

2. \[ 10000 \left( 1.0075 \right)^{120} \]

\[ P \cdot a^t \]
\[ a > 1 \]
\[ |a| < 1 \]