Math 2433
Exam 1 Review

Problem 1.
(a) Given the points $A(1, -2, 1), B(3, 2, 2), C(-2, 1, -5)$

(i) Prove that $A, B$ and $C$ are the vertices of a right triangle.

(ii) Determine the length of the hypotenuse of the triangle.

(iii) Find an equation for the sphere that has the hypotenuse for its diameter.

(iv) Find an equation for the plane that contains $A, B,$ and $C$.

(b) Given the vectors $a = 2i - k, \ b = 2i + 3j - k, \ c = -i + 3j - 2k$

(i) Calculate: $3(2a - b + c)$

(ii) Find: $\text{proj}_{b}c$

(iii) Determine a unit vector in the direction of $b \times a$.

(iv) Determine the volume of the parallelepiped that has $a, b, \text{and } c$ as sides.

Problem 2. Given the plane $P : 3x - y + 4z = 3$, the line $L : \frac{x - 1}{-2} = \frac{y + 4}{2} = \frac{z + 3}{2}$, and the point $A(-3, 0, 5)$:

(a) Determine whether $P$ and $L$ are parallel.

(b) Determine whether $A$ lies on $L$. If it does not, find the distance from $A$ to $L$.

(c) Determine the parametric equations for the line $M$ that passes through $A$ and is perpendicular to $P$.

(d) Find the point of intersection of $M$ and $P$.

Problem 3. Given the planes $P_1 : x + 4y - z = 10, \ P_2 : 3x - y + 2z = 4$, and the point $A(-2, 1, 0)$:

(a) Determine the equation for the plane that contains $A$ and is parallel to $P_1$.

(b) Determine whether $A$ lies on $P_2$. If it does not, find the distance from $A$ to $P_2$.

(c) Determine whether $P_1$ and $P_2$ are parallel. If not, find symmetric equations for the line of intersection of $P_1$ and $P_2$.

(d) Find the cosine of the angle between $P_1$ and $P_2$.

Problem 4.
(a) Let $f(t) = (2t + 1)i - \cos(\pi t)j$ and $g(t) = (t - 1)i + \cos(\pi t)j$

(i) Find $\lim_{t \to 3} f(t)$.

(ii) Calculate $[f(t) \cdot g(t)]'$.

(b) The position of an object at time $t$ is given by the vector function: $r(t) = (e^{-t} \sin t)i + (e^{-t} \cos t)j + 3tk$

Determine:

(i) The velocity vector: $v(t)$.

(ii) The speed of the object at time $t$.

(iii) The acceleration vector: $a(t)$.

(iv) $\lim_{t \to \infty} v(t)$
Problem 5.
(a) A curve $C$ in the plane is determined by the parametric equations:

$$
\begin{align*}
x(t) &= \frac{1}{2}t^2 + 1, \\
y(t) &= \frac{1}{3}t^3 - 1
\end{align*}
$$

(i) Find the length of $C$ from $t = 0$ to $t = 3$.
(ii) Calculate the curvature of $C$ at the point where $t = 1$.

(b) The vector function $\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$ determines a curve $C$ in space.

(i) Find scalar parametric equations for the tangent line to $C$ at the point where $t = 3$.
(ii) Find the length of $C$ from $t = 0$ to $t = 3$.
(iii) Determine the unit tangent vector $\mathbf{T}$ and the principal normal vector $\mathbf{N}$ at $t = 1$.
(iv) Find an equation for the osculating plane at $t = 1$.

Problem 6. The vector function $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} + \frac{1}{3}\sqrt{3}t^2\mathbf{k}$ determines a curve $C$ in space.

(a) Find the unit tangent vector $\mathbf{T}(t)$ and the principal normal vector $\mathbf{N}$.

(b) Find the curvature, $\kappa$ of $C$.

c) Determine the tangential and normal components of acceleration; express the acceleration vector, $\mathbf{a}(t)$ in terms of $\mathbf{T}$ and $\mathbf{N}$.

Problem 7.
(a) Let $f(x, y) = y^2e^{xy} + \frac{x}{y}$. Calculate $f_{xx}$ and $f_{yx}$.

(b) Let $z = \ln \sqrt{x^2 + y^2}$. Show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1$

c) Let $u = x^2 - 2y^2 + z^3$ where $x = \sin t$, $y = e^{2t}$, $z = 3t$. Calculate $\frac{du}{dt}$.

d) Let $z = e^{2x} \ln y$ where $x = u^2 - 2v$ and $y = v^2 - 2u$. Calculate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$

Problem 8. Let $f(x, y) = x^2 ye^{x-1} + 2xy^2$ and $F(x, y, z) = x^2 + 3yz + 4xy$.

(a) (i) Find the gradient of $f$.

(ii) Determine the direction in which $f$ decreases most rapidly at the point $(1, -1)$. At what rate is $f$ decreasing?

(b) (i) Find the gradient of $F$.

(ii) Find the directional derivative of $F$ at the point $(1, 1, -5)$ in the direction of the vector $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \sqrt{3}\mathbf{k}$.
(c) Find an equation for the tangent plane to the level surface \( F(x, y, z) = 3 \) at the point \((3, -1, -2)\).

(d) Find an equation for the tangent plane and scalar parametric equations for the normal line to the surface \( z = f(x, y) \) at the point \((1, -1, 1)\).

Problem 9. Given \( r(t) = 2 \sin t \mathbf{i} + 3 \cos t \mathbf{j}, g(t) = f(r(t)), \nabla f(0, 3) = 5 \mathbf{i} - 4 \mathbf{j} \) and \( \nabla f(2, 0) = -3 \mathbf{i} + 7 \mathbf{j} \), find:

(a) \( g'(0) \)

(b) \( g'(\pi/2) \)

Problem 10.

(a) Find the stationary points of \( f(x, y) = \frac{1}{4} x^3 + \frac{1}{2} y^2 + 2xy - 7x - y + 3 \).

(b) For each stationary point \( P \) found in (a), determine whether \( f \) has a local maximum, a local minimum, or a saddle point at \( P \).