MATH 1432

Section 13470
M-F 12:00 - 2:00pm SEC 103

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Office Hours:
2:00 – 4:00pm MW in 620 PGH or by appointment
\[
\int_1^4 (3 - |x-3|) \, dx = \\
= \int_1^4 3 \, dx - \int_1^4 |x-3| \, dx \\
= \left[ 3x \right]_1^4 - \left[ \int_1^3 (x-3) \, dx + \int_3^4 (x-3) \, dx \right] \\
= 3(4-1) - \left[ \frac{1}{2} (x-3)^2 \right]_1^4 \\
= 3(4-1) - \frac{1}{2} \left[ (4-3)^2 - (1-3)^2 \right] \\
= \frac{5}{2}
\]
Summary for the area between two curves:

Basic. curves.

\[ A = \int_a^b f(x) \, dx \]

Two non-intersecting curves.

\[ A = \int_a^b (f(x) - g(x)) \, dx \]
Two curves that intersect.

Set \( f(x) = g(x) \)

\[ \Rightarrow x = a, b \]

\[ A = \int_{a}^{b} [f(x) - g(x)] \, dx \]

Two curves that intersect several times.

\[ A = \int_{a}^{b} [f(x) - g(x)] \, dx + \int_{b}^{c} [g(x) - f(x)] \, dx \]

\[ A = \int_{a}^{c} |f(x) - g(x)| \, dx \]
More Examples:
Find the area bounded by the graphs of \( y = x^2 + 2, \ y = -x, \ x = 0 \) and \( x = 1 \).

\[
A = \int_0^1 \left[ x^2 + 2 - (-x) \right] \, dx \\
= \int_0^1 \left[ x^2 + x + 2 \right] \, dx \\
= \left[ \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x \right]_0^1 \\
= \frac{1}{3} + \frac{1}{2} + 2 = \frac{a+3+12}{6} \\
= \frac{17}{6}
\]
Find the area of the region bounded by $f(x) = 2 - x^2$ and $g(x) = x$.

Set \( f(x) = g(x) \)

\[
2 - x^2 = x
\]

\[
0 = x^3 + x - 2 = (x-1)(x+2)
\implies x = 1 \text{ or } x = -2
\]

\[
A = \int_{-2}^{1} [2 - x^2 - x] \, dx
\]

\[
= \left[ 2x - \frac{1}{3}x^3 - \frac{1}{2}x^2 \right]_{-2}^{1}
\]

\[
= \cdots = \frac{9}{2}
\]
Find the area bounded by \( f(x) = \sin x \) and \( g(x) = \cos x \) for \( x \in [0,2\pi] \)

\[
A = \int_0^{2\pi} |\sin(x) - \cos(x)| \, dx
\]

\[
= \int_0^{\frac{\pi}{4}} [\cos(x) - \sin(x)] \, dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} [\cos(x) - \sin(x)] \, dx + \int_{\frac{5\pi}{4}}^{2\pi} [\cos(x) - \sin(x)] \, dx
\]

\[
= \cdots = 4\sqrt{2}
\]
Find the average value of the function over the interval. 

\[ y = 2x + 3e^x, \quad [1, 4] \]

\[
\text{Avg. Value} = \frac{1}{4-1} \int_{1}^{4} [2x + 3e^x] \, dx
\]

\[ = \frac{1}{3} \left[ x^2 + 3e^x \right]_{1}^{4} \]

\[ = \frac{1}{3} \left[ 16 + 3e^4 - 1 - 3e \right] \]

\[ = \frac{1}{3} \left( 15 + 3e^4 - 3e \right) \]

\[ = 5 + e^4 - e \]
Set up the definite integral(s) that gives the area of the shaded region.

\[ y = (1 - x)(x - 3) \]

\[ y = 4x - x^2 \]

\[ A = \int_1^3 (1-x)(x-3) \, dx \]

\[ A = \int_0^4 [4x - x^2] \, dx \]
\[ y = x^2 - 4x + 7 \]
\[ y = 10 - 2x \]
\[ x^2 - 4x + 7 = 10 - 2x \]
\[ x^2 - 2x - 3 = 0 \]
\[ (x - 3)(x + 1) = 0 \]

\[ y = 6 - x \]
\[ y = \sqrt{x} \]
\[ (\sqrt{x})^2 = (6 - x)^2 \]
\[ x = 36 - 12x + x^2 \]
\[ 0 = x^2 - 13x + 36 \]
\[ 0 = (x - 4)(x - 9) \]
\[ x = 4 \text{ or } x = 9 \]

\[ A = \int_{0}^{3} [x^2 - 4x + 7] \, dx \]
\[ + \int_{3}^{5} [10 - 2x] \, dx \]

\[ A = \int_{0}^{4} \sqrt{x} \, dx + \int_{4}^{6} (6 - x) \, dx \]
Find the area bounded by \( f(x) = 3x^3 - x^2 - 10x \) and \( g(x) = -x^2 + 2x \).

First, find the intersection by setting the functions equal to each other. We get \( x = 0, -2, 2 \).

Next, determine which function is larger on each interval. What do you do if you don’t know how to graph the functions?

\[
\begin{align*}
3x^3 - x^2 - 10x &= -x^2 + 2x \\
3x^3 - 12x &= 0 \\
3x(x^2 - 4) &= 0 \\
3x(x - 2)(x + 2) &= 0
\end{align*}
\]
Finally, set up the integrals.

\[ A = \int_{-2}^{0} \left[ 3x^3 - x^3 - 10x - (-x^2 + 2x) \right] \, dx \]

\[ + \int_{0}^{2} \left[ -x^2 + 2x - (2x^3 - x^3 - 10x) \right] \, dx \]

\[ = \int_{-2}^{0} [3x^3 - 12x] \, dx + \int_{0}^{2} [12x - 3x^3] \, dx \]

\[ = \int_{-2}^{2} |f(x) - g(x)| \, dx \]
Now to change things up a bit…..

\( f(y) = -(y - 1)^2 + 1 \) and \( g(y) = -y \)

First, find the intersection by setting the functions equal to each other. We get \( y = 0, 3 \).

Set up the integrals.

\[
A = \int_{0}^{3} \left[ -(y-1)^2 + 1 - (-y) \right] dy
\]

When \( y = f(x) \) we integrate Top - Bottom

When \( x = g(y) \) we integrate Right - Left

\( x \) is a function of \( y \)
Find the area between the graphs of \( y = x + 6 \) and \( x = -y^2 \)

\[
A = \int_{-3}^{2} \left[-y^2 - (y-6)\right] \, dy
\]

\[
= \frac{125}{6}
\]
Find the area between the graph of \( f(x) = \begin{cases} 
    x^2 + 1 & \text{if } 0 \leq x \leq 1 \\
    3 - x & \text{if } 1 < x \leq 3
\end{cases} \) and the x-axis.

\[
A = \int_{0}^{1} (x^2 + 1) \, dx + \int_{1}^{3} (3 - x) \, dx
\]

\[
A = \ldots = \sqrt{\frac{10}{3}}
\]
Sketch the region bounded by the curves and find the area of that region.

\[ x = \sqrt{y}, \quad x - 2y = 0 \]

\[ x = \sqrt{y} \quad \text{and} \quad x = 2y \]

\[ \sqrt{y} = 2y \]

\[ y = 4y^2 \]

\[ 0 = 4y^2 - y = y(4y-1) \]

\[ y = 0 \quad \text{or} \quad y = \frac{1}{4} \]

\[ A = \int_{0}^{\frac{1}{4}} [\sqrt{y} - 2y] \, dy \]

\[ A = \int_{0}^{\frac{1}{4}} \left[ \frac{1}{2}x - x^2 \right] \, dx \]

\[ = \frac{1}{48} \]
\[ x = y^2, \quad x = 3 - 2y^2 \]

\[ 3 - 2y^2 = y^2 \]

\[ 3 = 3y^2 \quad \Rightarrow \quad 1 = y^2 \quad \Rightarrow \quad y = \pm 1 \]

\[ A = \int_{-1}^{1} [3 - 2y^2 - y^2] \, dy \]

\[ = \int_{-1}^{1} [3 - 3y^2] \, dy \]

\[ = 2 \int_{0}^{1} [3 - 3y^2] \, dy \]

\[ = 2 \left[ 3y - y^3 \right]_{0}^{1} \]

\[ = 2[3 - 1] = 4 \]
\[ y = |x|, \ 3y - x = 6 \implies y = \frac{1}{3}(x + 6) = \frac{1}{3}x + 2 \]

\[ A = \int_{-\frac{3}{2}}^{\frac{3}{2}} \left[ \frac{1}{3}x + 2 - 1 \times 1 \right] dx \]

\[ = \int_{-\frac{3}{2}}^{\frac{3}{2}} \left[ \frac{1}{3}x + 2 - (-x) \right] dx \]

\[ + \int_{0}^{3} \left[ \frac{1}{3}x + 2 - x \right] dx \]

\[ = \cdots = \frac{9}{2} \]
Let $R$ be the region in the first quadrant bounded by the graph of $y = 25 - x^2$ and the coordinate axes. Determine the value of $c$ such that $y = cx^2$ separates $R$ into two regions of equal area.

Set $cx^2 = 25 - x^2$

$(c+1)x^2 - 25 = 0$

$x^2 = \frac{25}{c+1}$

$x = \pm \sqrt{\frac{25}{c+1}} = \frac{5}{\sqrt{c+1}}$

Want $A_1 = A_2$

Total Area $= \int_0^5 (25-x^2) \, dx$

$= \frac{250}{3}$

Half of Total $= A_1 = A_2 = \frac{125}{3}$

\[
A_1 = \int_0^{\frac{5}{\sqrt{c+1}}} \left[ 25 - x^2 - cx^2 \right] \, dx
\]
\[ A_1 = \int_0^\sqrt{\frac{5}{c+1}} \left[ 25 - x^2 - \frac{c}{3} x^3 \right] \, dx \]

\[ = \left[ 25x - \frac{1}{3}x^3 - \frac{c}{9}x^3 \right]_0^{\sqrt{\frac{5}{c+1}}} \]

\[ = 25 \cdot \frac{\sqrt{5}}{\sqrt{c+1}} - \frac{1}{3} \left( \sqrt{c+1} \right)^3 \cdot \left( \frac{\sqrt{5}}{\sqrt{c+1}} \right)^3 \]

\[ = \frac{125}{\sqrt{c+1}} - \frac{1}{3} \frac{\sqrt{5}}{\sqrt{c+1}} \frac{125}{\sqrt{(c+1)^3}} \]

\[ = \frac{125}{\sqrt{c+1}} - \frac{1}{3} \frac{125}{\sqrt{c+1}} = \frac{2}{3} \cdot \frac{125}{\sqrt{c+1}} \]

Set \( A_1 = \frac{125}{3} = \frac{2}{3} \cdot \frac{125}{\sqrt{c+1}} \) \( \Rightarrow \) \( 1 = \frac{2}{\sqrt{c+1}} \)

\[ \Rightarrow \left( \sqrt{c+1} \right)^2 = (2)^2 \]

\[ c+1 = 4 \quad \Rightarrow \quad c = 3 \]
Use integration to find the area of the triangle whose vertices are $(0, 0), (1, 3)$ and $(1, 5)$. 
The function \( f(x) = x^3 + x \) is invertible. What is the area between the graph of \( y = f^{-1}(x) \) and the \( y \)-axis for \( 0 \leq y \leq 2 \)?
Section 7.4: Volumes of Known Cross Sections

* If the cross section is perpendicular to the x-axis and its area is a function of x, say \( A(x) \), then the volume of the solid from \( a \) to \( b \) is given by

\[
V = \int_{a}^{b} A(x) \, dx
\]

* If the cross section is perpendicular to the y-axis and its area is a function of y, say \( A(y) \), then the volume of the solid from \( c \) to \( d \) is given by

\[
V = \int_{c}^{d} A(y) \, dy
\]
1. Find the volume of the solid whose base is bounded by

\[ f(x) = 1 - \frac{1}{2}x, \quad g(x) = -1 + \frac{1}{2}x \quad \text{and} \quad x = 0 \]

if the solid is formed by squares perpendicular to the x-axis.

The side length is

\[ s(x) = f(x) - g(x) = 1 - \frac{1}{2}x - (-1 + \frac{1}{2}x) = 1 - \frac{1}{2}x + 1 - \frac{1}{2}x = 2 - x \]

So, \( A(x) = (s(x))^2 = (2-x)^2 \)

\[ V = \int_{0}^{2} A(x) \, dx = \int_{0}^{2} (2-x)^2 \, dx \]

\[ = -\frac{1}{3}(2-x)^3 \bigg|_{0}^{2} = \frac{1}{3} \left[ 2^3 - (2-2)^3 \right] = \frac{8}{3} \]
2. Find the volume of the solid whose base is bounded by
\[ f(x) = 1 - \frac{1}{2}x, \quad g(x) = -1 + \frac{1}{2}x \] and \( x = 0 \) if the solid is formed by
equilateral triangles perpendicular to the x-axis.

\[ b(x) = f(x) - g(x) = 2 - x \]
\[ A(x) = \frac{1}{2} b(x) \cdot h(x) \]
\[ = \frac{1}{2} b(x) \cdot \frac{\sqrt{3}}{2} b(x) \]
\[ = \frac{\sqrt{3}}{4} (b(x))^2 \]
\[ A(x) = \frac{\sqrt{3}}{4} (2-x)^2 \]
\[ V = \int_0^2 A(x) \, dx = \frac{\sqrt{3}}{4} \int_0^2 (2-x)^2 \, dx \]
\[ = \frac{\sqrt{3}}{4} \cdot \frac{8}{3} = \frac{2\sqrt{3}}{3} = \frac{2}{\sqrt{3}} \]
3. Find the volume of the solid whose base is bounded by \( f(x) = x^2, \quad g(x) = 8 - x^2 \) and the solid is formed by squares perpendicular to the x-axis.

\[ f(x) = g(x) \]

\[ x^2 = 8 - x^2 \]

\[ 2x^2 = 8 \]

\[ x^2 = 4 \]

\[ x = \pm 2 \]

\[ s(x) = 8 - x^2 - x^2 = 8 - 2x^2 \]

\[ A(x) = (s(x))^2 = (8 - 2x^2)^2 \]

\[ Vol = \int_{-2}^{2} A(x) \, dx = 2 \int_{0}^{2} (8 - 2x^2)^2 \, dx = 2 \cdot \int_{0}^{2} (64 - 32x^2 + 4x^4) \, dx \]

\[ = 2 \left[ 64x - \frac{32x^3}{3} + \frac{4x^5}{5} \right]_{0}^{2} = \frac{211}{15} \]
4. Find the volume of the solid whose base is bounded by \( y = \frac{1}{8}x^2 \) and \( y = 4 \) if the solid is formed by semicircles perpendicular to the y-axis.
5. Consider a solid whose base is the region inside the circle $x^2 + y^2 = 4$. If cross sections taken perpendicular to the x-axis are squares, find the volume of this solid.
Volume with the Disc Method:

Revolving about the x-axis:  $$V = \int_{a}^{b} \pi \left[ f(x) \right]^2 \, dx$$
Revolving about the y-axis: \[ V = \int_c^d \pi \left[ g(y) \right]^2 \, dy \]
Let R be the region bounded by the x-axis and the graphs of \( y = \sqrt{x} \) and \( x = 4 \). Sketch and shade the region R. Label points on the x and y-axis.

a. Give the formula the area of region R

b. Find the area of region R
c. Give the formula the volume of the solid generated when the region R is rotated about the x-axis.
d. Find the volume for the solid in (c).
Let R be the region bounded by the y-axis and the graphs of \( y = \sqrt{x} \) and \( y = 2 \). Sketch and shade the region R. Label points on the x and y-axis.
Give the formula the volume of the solid generated when the region R is rotated about the y-axis.
Find the volume for the solid.