RESEARCH STATEMENT

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1. Ergodic Theory

Measures are a generalization of length or volume. As in calculus, a measure defines an integral of a real-valued function, via a limiting process, as the average value of the function over the space. Ergodic theory connects this average value over a space with the long-term behavior of points (the average value under iteration).

A standard assumption in ergodic theory is that a transformation on a space preserves the measure on the space. This is like the standard assumption in fluid dynamics that water is incompressible. In this setting, an isomorphism between spaces should preserve the measure of sets.

As a weaker assumption, a transformation might change the measure of some sets, but still preserve sets of measure zero (a nonsingular system). This is like using fluid dynamics to study the flow of air, which acts like a compressible fluid. As in a nonsingular system, the volume of a container of air decreases when it is compressed, but the air cannot be compressed to have no volume. In this weaker setting, the appropriate notion of isomorphism is called orbit equivalence.

In [Fur13, Fur14a, Fur14b, Fur15], I examine the ergodic properties and orbit equivalence of transformations on the p-adic numbers with respect to independent and identically distributed (i.i.d.) product measures. Study of the p-adic numbers began in number theory, and i.i.d. product measures are related to the distribution of digits. From this point of view, the combination is a natural one to study. From an ergodic point of view, however, the interactions turn out to be surprisingly complicated, even for such a basic transformation as a translation on the p-adic numbers.

1.1. Definitions.

Fix a prime number p. The set of p-adic integers is a set of formal power series in the prime p:

$$\mathbb{Z}_p = \left\{ \sum_{i=0}^{\infty} a_i p^i : a_i \in \mathbb{Z} \text{ and } 0 \leq a_i < p \right\}.$$

If a is a nonzero p-adic number, then $\operatorname{ord}_p(a) = \min\{i : a_i \neq 0\}$ is defined. Then the p-adic absolute value is defined by

$$|a|_p = \begin{cases} p^{-\operatorname{ord}_p(a)} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0. \end{cases}$$

This absolute value induces a metric topology with a basis of balls

$$B_{p^n}(a) = \{x \in \mathbb{Z}_p : |x - a|_p \leq p^n\} \text{ with } n \in \mathbb{Z} \text{ and } a \in \mathbb{Z}_p\,.$$

These balls are both open and closed, making the topology totally disconnected. Under this topology, $\mathbb{Z}_p$ is topologically isomorphic to the one-sided product space on p symbols. In fact, balls in $\mathbb{Z}_p$ correspond to cylinder sets in the product space.

Following the natural analogy with product spaces, we define independent and identically distributed (i.i.d.) product measures on $\mathbb{Z}_p$. If $(q_0, q_1, \ldots, q_{p-1})$ is a probability vector, then $\mu = \prod \{q_0, q_1, \ldots, q_{p-1}\}$ is the i.i.d. product measure. Under this measure, a ball centered at $a \in \mathbb{Z}_p$ of radius p^n has measure

$$\mu(B_{p^n}(a)) = \prod_{i=0}^{n-1} q_{a_i}.$$

In particular, Haar measure (the unique translation-invariant probability measure on $\mathbb{Z}_p$) is the i.i.d. product measure $\lambda = \prod \{1/p, 1/p, \ldots, 1/p\}$.

The standard odometer (also called the adding machine) on the product space corresponds to translation by 1 on $\mathbb{Z}_p$ ($T_1(x) = x + 1$). More generally, addition and multiplication on $\mathbb{Z}_p$ is defined coordinate-wise with carries. The odometer on a product space is a classical and well-understood example in ergodic theory. Looking at the example as translation by 1 on $\mathbb{Z}_p$, it is natural to extend the investigation to translation by other elements, $T_a(x) = x + a$ for $a \in \mathbb{Z}_p$.

1.2. **Ergodic Properties.** A transformation is nonsingular with respect to a measure if it preserves sets of measure 0. Translation by 1 on $\mathbb{Z}_p$ is isomorphic to the odometer on the product space on p symbols, which is well-known to be nonsingular with respect to i.i.d. product measures. Since translation by an integer is an iterate of translation by 1, translation by $n \in \mathbb{Z} \subset \mathbb{Z}_p$ is nonsingular with respect to i.i.d. product measures. Also, Haar measure is translation-invariant, so all translations are nonsingular with respect to Haar measure. Outside of these easy cases, singular behavior can occur, as proved in [Fur14b].

Theorem 1.1. *Let $a \in \mathbb{Z}_p$ be a rational number but not an integer. Then T_a is singular with respect to any i.i.d. product measure that is not Haar measure.*

Whether translation by $a \in \mathbb{Z}_p \setminus \mathbb{Q}$ is singular or nonsingular with respect to i.i.d. product measures other than Haar measure is still an open question.

1.3. **Convergence of Cyclic Approximations.** For a fixed $a \in \mathbb{Z}_p$, define $S_n : \mathbb{Z}_p \to \mathbb{Z}_p$ by

$$(S_n(x))_i = \begin{cases} (T_a(x))_i & \text{if } 0 \leq i < n \\ x_i & \text{if } i \geq n. \end{cases} \tag{1}$$

Then $\{S_n\}$ is a sequence of cyclic approximations of T_a. The results in [Fur15] describe the convergence of these cyclic transformations to T_a, use the approximations to describe the ergodic and spectral properties of T_a (and of transitive isometries on $\mathbb{Z}_p$ more generally), and relate the Euclidean algorithm to a directed graph description of the cyclic approximations of T_a.

In particular, a sequence of transformations $s_n(x)$ on a metric space (X, d) converges to $t(x)$ *uniformly in* x if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(s_n(x), t(x)) < \epsilon$ for all $n \geq N$ and all $x \in X$. It follows from the definition of S_n that S_n converges to T uniformly in x on $\mathbb{Z}_p$.

In contrast, consider convergence in the strong topology. A sequence of transformations s_n on a measure space $(X, \mathcal{A}, \mu)$ *converges in the strong topology* on the set of transformations to a transformation t on $(X, \mathcal{A}, \mu)$ if

$$\mu \{x \in X : s_n(x) \neq t(x)\} \to 0 \text{ as } n \to \infty.$$

In particular, translation by $a \in \mathbb{Z}_p \setminus \mathbb{Z}$ with respect to an i.i.d. product measure is an example where the approximations do not converge in the strong topology.

Theorem 1.2. *Let $a \in \mathbb{Z}_p^\times$ and let μ be an i.i.d. product measure. For $n \in \mathbb{N}$, define S_n by Equation (1). Then S_n converges in the strong topology to T_a if and only if $a \in \mathbb{Z} \subset \mathbb{Z}_p$.*

1.4. **Orbit Equivalence Classes.** Two invertible, nonsingular, and ergodic transformations, T on $(X, \mathcal{A}, \mu)$ and S on $(Y, \mathcal{B}, \nu)$, are orbit equivalent if there exists a bimeasurable map Φ such that $\mu \circ \Phi^{-1}$ and ν are mutually absolutely continuous and for μ-almost every $x \in X$,

$$\{\Phi(T^n x) : n \in \mathbb{Z}\} = \{S^m(\Phi x) : m \in \mathbb{Z}\}.$$

The paper [Fur14a] contains examples of p-adic transformations in different orbit equivalence classes and looks at the behavior of orbit equivalence classes under iteration.

If two measure-preserving transformations are isomorphic, then their iterates are also isomorphic. In contrast,

Theorem 1.3. *There exist invertible, nonsingular, and ergodic systems $(X, \mathcal{B}, \mu; T)$ and $(Y, \mathcal{C}, \nu; S)$ that are orbit equivalent, but the iterates $(X, \mathcal{B}, \mu; T^2)$ and $(Y, \mathcal{C}, \nu; S^2)$ are not orbit equivalent.*

2. Size of p-adic Julia Sets

Does a small error in measurement ruin predictions for the future? For a transformation on a set X, the set X is partitioned into two subsets, the Fatou set and the Julia set. In the Fatou set, if two points are near each other, then they stay relatively near each other under iteration by the transformation—a small error stays relatively small. In contrast, if two points in the Julia set are near each other, they may have drastically different behavior and end up far apart under iteration—small errors can become very large.

For polynomial functions over the p-adic numbers, topological descriptions of the Julia sets for early examples appear in [Hsi96, Sil07, WS98]. Let f be a polynomial or rational function on a compact-open subset X of K, a finite extension of $\mathbb{Q}_p$. Under some additional assumptions on f, Fan, Liao, Wang, and Zhou use incidence matrices and a subshift in [FLWZ07] to topologically describe the the dynamics of f on the Julia set defined by $\mathcal{J}(f, X) = \bigcap_{j=0}^{\infty} f^{-j}(X)$.

Following the setting in [FLWZ07], my research in [Fur16] examines the Haar measure and Hausdorff dimension of balls in a finite extension of $\mathbb{Q}_p$ and then describes the size of p-adic Julia sets using Haar measure and Hausdorff dimension.

Theorem 2.1. *Let K be an extension of $\mathbb{Q}_p$ of degree n, and let λ be Haar measure on K. Let $f : X \to K$ be a map on a compact-open $X \subset K$ such that $X = \bigcup_{i=1}^{k} B_{p^{r_1}}(c_i)$, where*
(a) $f^{-1}(X) \subset X$,
(b) $f(B_{p^{r_1}}(c_i)) \bigcap X \neq \emptyset$ for all $1 \leq i \leq k$,
(c) $\max_{1\leq i\leq k} \{|c_i|_p\} = p^{r_2}$ for some $r_2 > r_1$, and
(d) for all $1 \leq i \leq k$, $|f(x) - f(y)|_p = p^{r_2 - r_1}|x - y|_p$ for all $x, y \in B_{p^{r_1}}(c_i)$.

If $k = p^{n(r_2 - r_1)}$, then $\lambda(\mathcal{J}(f, X)) = p^{nr_2}$ and $Hdim(\mathcal{J}(f, X)) = n$.
If $k < p^{n(r_2 - r_1)}$, then $\mathcal{J}(f, X) \subsetneq X$, $\lambda(\mathcal{J}(f, X)) = 0$, and

$$Hdim(\mathcal{J}(f, X)) = \frac{\ln(k)}{(r_2 - r_1)\ln(p)}.$$

Finally, [Fur16] gives a large class of polynomials that satisfy the hypotheses of Theorem 2.1. Among these polynomials are the standard examples from [Hsi96, Sil07, WS98] and examples that demonstrate the dependence of the definition of $\mathcal{J}(f, X)$ on the choice of finite extension.

Recall that you can think of the Julia set as the region where a small error in the input might give a large error in the output of your system. Similarly, you can think of a Mandelbrot set or bifurcation locus as a region where a small error in a parameter describing your system might give an overall system with drastically different properties.

In dynamics on the p-adic numbers, interest has only recently shifted from looking at the Julia set for a single transformation to looking at the Mandelbrot set for a family of polynomials [Jon07, And13, Sil07]. In complex dynamics, the Mandelbrot set has many interpretations and relationships with the dynamics of the family. One of my research goals is to understand whether and how these interpretations apply to p-adic Mandelbrot sets. A better understanding of the relationships between the p-adic Mandelbrot set and the dynamics of the families may illuminate the shape of the p-adic Mandelbrot set, which generally has a complicated shape when the degree of the polynomial family is larger than the prime p.

3. Collaborations

In [CF16], David Constantine and I examine one-sided ergodic Hilbert transforms $\sum_{n=1}^{\infty} f \circ T^n(x)/n$, where T is an irrational circle rotation and f is a mean-zero function that is equal to ± 1 on finite unions of intervals. We show that there are Liouville rotations for which the transform diverges everywhere, which is stronger than previous results that conclude almost-everywhere divergence. We give a new proof that the one-sided ergodic Hilbert transform converges everywhere if the rotation is non-Liouville. This proof technique allows us to prove a new result—there exist Liouville rotations for which the transform converges.

Daniel Cuzzocreo, Scott Kaschner, and I are studying bifurcation loci in two-parameter families of rational maps. The bifurcation locus is the set of unstable parameters where the motion of the Julia set is discontinuous. We are studying how the bifurcation locus in one parameter plane changes as we vary the other parameter. We are looking at relationships between critical points (looser than strict orbit relations) to explain unexpected symmetry in the bifurcation locus, as in Question 2.4 of De Marco, Wang, and Ye's paper [DMWY15].

As a generalization of [Haw03], Jane Hawkins, Lorelei Koss, and I are examining the families $f_a(z) = a(z+1)^d/z^{d-k}$ for $d \geq 2$ and $1 \leq k \leq d-1$. We are examining the parameter spaces, the connectivity and structure of the Julia sets, relationships between the families for different values of d and k, and possible limiting dynamics as $d \to \infty$.

James Keesling and I are investigating locally complicated spaces. These spaces are path connected, locally path connected, but not necessarily semi-locally simply connected. In the absence of standard covering space theory, we are examining generalized covering spaces and topologized fundamental groups. The general theory in this area so far is quite abstract. We use ultrafilters and inverse limits to give more concrete and intuitive constructions and examples.

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