Conservative semi-Lagrangian finite difference WENO formulations with applications to the Vlasov equation

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Abstract

In this paper, we propose a new conservative semi-Lagrangian (SL) finite difference (FD) WENO scheme for linear advection equations, which can serve as a base scheme for the Vlasov equation by Strang splitting [4]. The reconstruction procedure in the proposed SL FD scheme is the same as the one used in the SL finite volume (FV) WENO scheme [3]. However, instead of inputting cell averages and approximate the integral form of the equation in a FV scheme, we input point values and approximate the differential form of equation in a FD spirit, yet retaining very high order (fifth order in our experiment) spatial accuracy. The advantage of using point values, rather than cell averages, is to avoid the second order spatial error, due to the shearing in velocity ($v$) and electrical field ($E$) over a cell when performing the Strang splitting to the Vlasov equation. As a result, the proposed scheme has very high spatial accuracy, compared with second order spatial accuracy for Strang split SL FV scheme for solving the Vlasov-Poisson (VP) system. We perform numerical experiments on linear advection, rigid body rotation problem; and on the Landau damping and two-stream instabilities by solving the VP system. For comparison, we also apply (1) the conservative SL FD WENO scheme, proposed in [21] for incompressible advection problem, (2) the conservative SL FD WENO scheme proposed in [20] and (3) the non-conservative version of the SL FD WENO scheme in [3] to the same test problems. The performances of different schemes are compared by the error table, solution resolution of sharp interface, and by tracking the conservation of physical norms, energies and entropies, which should be physically preserved.

Keywords: semi-Lagrangian methods; finite difference/finite volume scheme; conservative scheme; WENO reconstruction; Vlasov equation; Landau damping; two-stream instability.
1 Introduction

In this paper, we propose a new conservative semi-Lagrangian (SL) finite difference (FD) weighted essentially non-oscillatory (WENO) scheme, by utilizing the same reconstruction procedure as the one used in the SL finite volume (FV) WENO scheme in [3] for solving the 1-D advection equation

\[ u_t + cu_x = 0, \quad \text{where} \quad c \quad \text{is a constant.} \quad (1.1) \]

This work is motivated by the kinetic plasma applications, where the Vlasov equation is often numerically solved by the following procedure. First, the Strang splitting is applied to decouple the high-dimensional nonlinear Vlasov equation into a sequence of linear advection equations, such as (1.1); then a SL scheme is applied to solve those decoupled 1-D equations. The SL approach for solving the Strang split Vlasov equation has been very popular in the plasma simulation community, see for example [25, 12, 28, 3, 20, 9], as the scheme for the split 1-D equation is usually simple, effective and free of CFL condition, which is a restriction in an Eulerian approach. There are many variants in designing a SL scheme. Specifically, we characterize a SL scheme by the following three key components.

1. A solution space. The solution space can be point values, integrated mass (cell averages), or a piecewise polynomial function living on a fixed numerical grid, corresponding to the SL FD scheme [3, 20, 15], SL FV scheme [12, 9] and the characteristic Galerkin method [5, 17] respectively.

2. Propagation. In each of the time step evolution, information is propagated along characteristics. Usually, a high order interpolation or reconstruction procedure, which determines the spatial accuracy of the scheme, is applied to recover the information among discrete information on the solution space. In the literature, there are a variety of interpolation/reconstruction choices, such as the piecewise parabolic method (PPM) [7], positive and flux conservative method (PFC) [13], spline interpolation [8], cubic interpolation propagation (CIP) [26], ENO/WENO interpolation or reconstruction [16, 24, 3, 20, 21]. We refer to [10, 27, 9] for the comparison of different reconstruction procedures.
3. **Projection.** Lastly, the evolved solution is projected back onto the solution space, updating the numerical solution at $t^{n+1}$.

It is known that the mass conservation is a very important property of a SL scheme. Failure to conserve the mass might lead to instability of the scheme [15]. To conserve the mass, a scheme working with integrated mass in a FV spirit, seems more natural and straightforward [9]. On the other hand, we argue that it is advantageous to work with point values (FD scheme), rather than cell averages (FV scheme), due to the shearing of advection coefficients ($v$ and $E$) over a cell, in the context of Strang splitting for the Vlasov equation or other kinetic equations of similar kind. Such shearing would reduce the spatial accuracy of a SL FV scheme to second order, while a SL FD scheme can retain the same high order accuracy as the reconstruction. Due to above considerations, a conservative scheme that works with point values seems ideal. Such schemes have been constructed in [20, 21]. In this paper, we propose another approach of designing a conservative SL FD scheme, by utilizing the same WENO reconstruction procedure as in a SL FV scheme [3]. The idea is motivated by the very close relationship between the FV and FD WENO schemes [16, 24] for a semi-discrete equation. The scheme designed in this paper is easier to implement than those in [20, 21]. We compare the performance of the proposed scheme with the conservative or non-conservative SL FD WENO schemes designed earlier in [20, 21, 3]. We also note that the scheme designed in [21] was only applied to advection in an incompressible flow in [21]; while it is applied to the Vlasov-Poisson (VP) system for the first time in this paper, in the context of comparison with other SL FD schemes.

The paper is organized as follows. Section 2 is a review of semi-discrete FV and FD WENO schemes for advection equations. Section 3 presents the SL FV WENO scheme [3], and the proposed conservative SL FD WENO scheme, based on the same WENO reconstruction procedure as the one used in the SL FV WENO scheme. Section 4 compares the proposed scheme with the conservative SL FD WENO scheme in [20, 21] and the non-conservative SL FD WENO scheme in [3] for linear advection and rigid body rotation. Section 5 demonstrates the performance of the proposed schemes, in comparison with other SL FD WENO schemes [20, 21, 3], through the classical Landau damping and two stream instabilities by solving the VP system. Section 6 gives the conclusion.
2 FV and FD WENO methods

In this section, we will briefly review the FV and FD WENO spatial discretization for a semi-discrete 1-D linear advection equation

\[ u_t + (cu)_x = 0, \quad \text{on} \quad [a, b], \quad (2.1) \]

with the initial condition \( u(x, t = 0) = u_0(x) \). For simplicity, we assume a periodic boundary condition. The purpose of this review section is to recall the WENO reconstruction procedures in the FV and FD schemes, serving as a preparation for establishing the very close relationship of the reconstruction procedures in SL FV and FD schemes in Section 3. The readers are referred to [6, 24] for more details. In this paper, we adopt the following spatial discretization of the domain \([a, b]\)

\[ a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b, \quad (2.2) \]

where \( I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], \, i = 1, \cdots, N \) are uniform numerical cells with centers \( x_i = \frac{1}{2}(x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}}) \) and cell sizes \( \Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = (b-a)/N \). We use \( \bar{u}_i = \frac{1}{\Delta x} \int_{x_i} u(\xi, t) d\xi \) to denote the cell averages of the solution over \( I_i \) and use \( u_i = u(x_i, t) \) to denote the point value of the solution at \( x = x_i \). We use \( \bar{u}_i^n \) and \( u_i^n \) to denote the cell average/point value of the solution over \( I_i \) at \( x_i \) at time \( t = t^n \) respectively.

2.1 FV formulation

The FV scheme evolves the cell averages of the solution \( \bar{u}_i, \, i = 1, \cdots, N \), by approximating the integral form of the equation (2.1)

\[ \frac{d}{dt} \bar{u}_i = -\frac{1}{\Delta x}(\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}), \quad \forall i = 1, \cdots, N, \quad (2.3) \]

where the numerical flux

\[ \hat{f}_{i+\frac{1}{2}} = \hat{f}(u^-_{i+1/2}, u^+_{i+1/2}) \]

is consistent with the physical flux \( f(u) = cu \). It is Lipschitz continuous and monotonically increasing/decreasing with respect to the first/second argument. For example, an upwind flux would be \( \hat{f}_{i+\frac{1}{2}} = cu^-_{i+1/2} \), if \( c > 0 \); and \( \hat{f}_{i+\frac{1}{2}} = cu^+_{i+1/2} \) otherwise. The values of \( u^\pm_{i+1/2} \)
can be reconstructed in a WENO fashion from the cell averages in a neighborhood stencil \( \{u_{i-p}, \cdots, u_{i+q}\} \). Specifically, one more point from the left \((p = q)\) will be taken to reconstruct \( u_{i+1/2}^- \), and one more point from the right \((p = q - 2)\) will be taken to reconstruct \( u_{i+1/2}^+ \). We refer to [6, 24] for the details of WENO reconstructions. In the method of line (MOL) procedure, the time derivative on the L.H.S of equation (2.3) is discretized by a stable time integrator, such as the third order strong stability preserving (SSP) Runge-Kutta (RK) method [14]. There are other types of time discretization available in the literature, e.g. the Lax-Wendroff type in [22].

### 2.2 FD formulation

The FD scheme evolves the point values of the solution \( u_i, i = 1, \cdots, N \), by approximating the equation (2.1) directly. The scheme is of conservative form

\[
\frac{d}{dt} u_i = -\frac{1}{\Delta x}(\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}).
\]  

(2.4)

To obtain a high order approximation, a sliding average function \( h(x) \) is introduced, such that

\[
\frac{1}{\Delta x} \int_{x-\Delta x/2}^{x+\Delta x/2} h(\xi) d\xi = cu(x, t).
\]  

(2.5)

Taking the \( x \) derivative of the above equation gives

\[
\frac{1}{\Delta x} \left( h \left( x + \frac{\Delta x}{2} \right) - h \left( x - \frac{\Delta x}{2} \right) \right) = (cu)_x.
\]  

(2.6)

Therefore the numerical flux \( \hat{f}_{i+\frac{1}{2}} \) in equation (2.4) can be taken as \( h(x_{i+\frac{1}{2}}) \), which can be reconstructed from neighboring cell averages of \( h(x) \), \( \tilde{h}_j = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} h(\xi) d\xi \) \((2.5)\) \( = cu(x_j, t) \), \( j = i-p, \cdots, i+q \), by the WENO reconstruction [24]. The stencil \( \{u_{i-p}, \cdots, u_{i+q}\} \) is chosen to be upwind biased. Specifically, when \( c \geq 0 \), one more point from the left \((p = q)\) will be taken to reconstruct \( \hat{f}_{i+\frac{1}{2}}^- \); when \( c < 0 \), one more point from the right \((p = q - 2)\) will be taken. Similar to equation (2.3), equation (2.4) is further discretized in time by a stable time integrator, such as the third order SSP RK method [14].

**Remark 2.1.** (About WENO reconstructions) Although FV and FD schemes are working with different quantities of the solution (cell averages and point values respectively), the
WENO reconstruction procedure in the FV and FD schemes is the same. Specifically, the
WENO reconstruction in both FV and FD schemes can be thought of as a blackbox, whose
input consists of cell averages of a given function, and whose output consists of highly
accurate point values of the same function at cell boundaries. The only difference is that
in the FV scheme, the reconstruction procedure works with the unknown function \( u \) itself;
but in the FD scheme, the reconstruction procedure works with a sliding average function \( h \)
defined in equation (2.5). In fact, for the linear advection equation with constant coefficients,
the numerical procedure of the FV and FD WENO schemes is exactly the same. The only
difference between these two schemes in this case is in the initial condition (the FV scheme
uses the cell averages of the initial condition while the FD scheme uses its point values).

3 The SL FV and FD WENO schemes

In this section, we will start with a description of the SL FV WENO scheme, originally
introduced in [3]. We will briefly outline the scheme in Section 3.1, which will be closely
related to the SL FD WENO scheme introduced in Section 3.2.

3.1 SL FV WENO scheme

In a FV scheme, it is the cell averages of the solution \( \bar{u}_i \) that are being updated. A SL FV
scheme is formulated based on the following observation: at each of the cell boundaries at
time level \( t^{n+1} \), say \( (x_{i+\frac{1}{2}}, t^{n+1}) \), there exists a backward characteristic line, denoted as \( \Gamma_{i+\frac{1}{2}} \),
with its foot located on time level \( t^n \) at \( y_{i+\frac{1}{2}} \). Since there is no flux passing through the
characteristics lines, due to the mass conservation, the cell averages of the solution can be
updated by

\[
\bar{u}_{i+\frac{1}{2}}^{n+1} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, t^{n+1}) dx = \frac{1}{\Delta x} \int_{y_{i-\frac{1}{2}}}^{y_{i+\frac{1}{2}}} u(x, t^n) dx. 
\]

To evaluate the R.H.S of equation (3.1), we would like to locate \( y_{i\pm\frac{1}{2}} \) and to reconstruct the
integral by the given cell averages \( \{\bar{u}_i^n\}_{i=1}^N \). For a linear advection equation, the characteristics
are straight lines with slope \( 1/c \), therefore \( y_{i\pm\frac{1}{2}} = x_{i\pm\frac{1}{2}} - c\Delta t \); and the integral can be
reconstructed by a WENO interpolation of the primitive function of \( u \). More specifically, we
let \( U^n(x) = \int x u(\xi, t^n) d\xi \) to be the primitive function of \( u(x, t^n) \) with \( U(a) = 0 \), then
\[
U^n_i = U^n(x_i) = \Delta x \sum_{j=1}^{i} \bar{a}_j^n, \quad \text{and} \quad \frac{1}{\Delta x} \int_{y_{i-1/2}}^{y_{i+1/2}} u(x, t^n) dx = \frac{1}{\Delta x} (U^n(y_{i+1/2}) - U^n(y_{i-1/2})).
\]

We reconstruct the point values of \( U \), e.g. \( U(y_{i+1/2}) \), from the grid point values of \( \{U^n_i\}_{i=1}^{N} \) by a WENO interpolation. The WENO interpolation algorithm has been discussed in [23, 2, 19]. The difference between the WENO interpolation in this SL FV framework and that in the literature, e.g. in [23], is that (i) to obtain a \((2k+1)\)th order local truncation error, we need a \((2k+2)\)-point stencil due to the \(\frac{1}{\Delta x}\) factor in equation (3.1), and (ii) the smoothness indicators in the WENO algorithm should be derived, taking into account of second and higher order derivatives of \( U(x) \), rather than first and higher derivative of \( u(x) \) as in the WENO interpolation in [23].

In the following, we provide a sixth order WENO interpolation, giving a fifth order SL FV WENO scheme, as used in our simulations in Section 4 and 5. The goal is to construct \( U(x) \) for any \( x \in [x_{i-1}, x_i] \) (or \( \xi = \frac{x-x_i}{\Delta x} \in [-1, 0] \)) in the WENO fashion from a 6-point stencil \( S = \{U_{i-3}, U_{i-2}, U_{i-1}, U_i, U_{i+1}, U_{i+2}\} \), which can be decomposed into three 4-point stencils
\[
S_1 = \{U_{i-3}, U_{i-2}, U_{i-1}, U_i\}, \quad S_2 = \{U_{i-2}, U_{i-1}, U_i, U_{i+1}\}, \quad S_3 = \{U_{i-1}, U_i, U_{i+1}, U_{i+2}\}.
\]

**Linear weight function** \( C_k(\xi) \), \( k = 1, 2, 3 \). We denote \( Q(\xi) \) as the polynomial of degree 5 interpolating the point values in the 6-point stencil \( S \) and denote \( P_k(\xi) \) as the polynomial of degree 3 interpolating the point values in the stencil \( S_k \). It is proved in [2] that there exist linear weights \( C_k(x) \), which are actually polynomials of degree 2, such that
\[
Q(x) = \sum_{k=1}^{3} C_k(x) P_k(x).
\]

In the sixth order case of our implementation, we have
\[
Q(\xi) = (U_{i-3}, U_{i-2}, U_{i-1}, U_i, U_{i+1}, U_{i+2}) \begin{pmatrix} 0 & -1/30 & 0 & 1/24 & 0 & -1/120 \\ 0 & 1/4 & -1/24 & -7/24 & 1/24 & 1/24 \\ 0 & -1 & 2/3 & 7/12 & -1/6 & -1/12 \\ 1 & 1/3 & -5/4 & -5/12 & 1/4 & 1/12 \\ 0 & 1/2 & 2/3 & 1/24 & -1/6 & -1/24 \\ 0 & -1/20 & -1/24 & 1/24 & 1/24 & 1/120 \end{pmatrix} \begin{pmatrix} 1 \\ \xi \\ \xi^2 \\ \xi^3 \\ \xi^4 \\ \xi^5 \end{pmatrix}
\]
Therefore we have

\[ P_1(\xi) = (U_{i-3}, U_{i-2}, U_{i-1}, U_i) \begin{pmatrix} 0 & -1/3 & -1/2 & -1/6 \\ 0 & 3/2 & 2 & 1/2 \\ 0 & -3 & -5/2 & -1/2 \\ 1 & 11/6 & 1 & 1/6 \end{pmatrix} \begin{pmatrix} 1 \\ \xi \\ \xi^2 \\ \xi^3 \end{pmatrix} \]

\[ P_2(\xi) = (U_{i-2}, U_{i-1}, U_i, U_{i+1}) \begin{pmatrix} 0 & 1/6 & 0 & -1/6 \\ 0 & -1 & 1/2 & 1/2 \\ 1 & 1/2 & -1 & -1/2 \\ 0 & 1/3 & 1/2 & 1/6 \end{pmatrix} \begin{pmatrix} 1 \\ \xi \\ \xi^2 \\ \xi^3 \end{pmatrix} \]

\[ P_3(\xi) = (U_{i-1}, U_i, U_{i+1}, U_{i+2}) \begin{pmatrix} 0 & -1/3 & 1/2 & -1/6 \\ 1 & -1/2 & -1 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \\ 0 & -1/6 & 0 & 1/6 \end{pmatrix} \begin{pmatrix} 1 \\ \xi \\ \xi^2 \\ \xi^3 \end{pmatrix} \]

\[ C_1(\xi) = \frac{1}{20}(\xi - 1)(\xi - 2), \quad C_2(\xi) = -\frac{1}{10}(\xi + 3)(\xi - 2), \quad C_3(\xi) = \frac{1}{20}(\xi + 3)(\xi + 2). \]

**Nonlinear weights by the smoothness indicator.** The smoothness indicators \( \beta_k \) are determined by

\[ \beta_k = \sum_{l=2}^{3} \int_{-1}^{0} \left( \frac{d}{d\xi} P_k(\xi) \right)^2 d\xi. \quad (3.2) \]

Therefore we have

\[ \beta_1 = -9 U_{i-3} U_{i-2} + 4/3 U_{i-3}^2 - 11/3 U_{i-3} U_i + 10 U_{i-3} U_{i-1} + 14 U_{i-2} U_i \\
+ 22 U_{i-1}^2 - 17 U_{i-1} U_i + 10/3 U_i^2 + 16 U_{i-2}^2 - 37 U_{i-2} U_{i-1} \]

\[ \beta_2 = -7 U_{i-2} U_{i-1} + 4/3 U_{i-2}^2 - 5/3 U_{i-2} U_{i+1} + 6 U_{i-2} U_i + 6 U_{i-1} U_{i+1} \\
+ 10 U_i^2 - 7 U_i U_{i+1} + 4/3 U_{i+1}^2 + 10 U_{i-1}^2 - 19 U_{i-1} U_i \]

\[ \beta_3 = -17 U_{i-1} U_i + 10/3 U_{i-1}^2 - 11/3 U_{i-1} U_{i+2} + 14 U_{i-1} U_{i+1} + 10 U_i U_{i+2} \\
+ 16 U_{i+1}^2 - 9 U_{i+1} U_{i+2} + 4/3 U_{i+2}^2 + 22 U_i^2 - 37 U_i U_{i+1}. \]

The nonlinear weights are chosen to be

\[ w_k(\xi) = \frac{\tilde{w}_k(\xi)}{\sum_{k=1}^{3} \tilde{w}_k(\xi)}; \quad \tilde{w}_k(\xi) = \frac{C_k(\xi)}{(\epsilon + \beta_k)^2}, \]

where \( \epsilon \) is chosen to be \( 10^{-6} \) in our simulations.
3.2 SL FD WENO scheme

In a FD scheme, it is the point values of the solution \( u_i \) that is being updated. There have been two different formulations of conservative SL FD WENO schemes in the literature \([20, 21]\). The one in \([21]\) is advantageous over that in \([20]\) in that the formulation in \([21]\) can be applied to equations with variable coefficients; while the one in \([20]\), based on the splitting of interpolation matrices, only applies to advection with constant coefficients. In this paper, we introduce another formulation of the SL FD WENO scheme, based on the SL FV WENO scheme formulated in Section 3.1. Specifically, we show that instead of working with cell averages of the solution in a SL FV WENO scheme, if we work with point values in a FD formulation using the same WENO reconstruction procedure, we retain high order accuracy (fifth order in our simulations). This new SL FD WENO scheme is simpler to implement than that in \([21]\). Unfortunately, as the SL FD WENO scheme in \([20]\), our current formulation only applies to equations with constant coefficients.

It is known that at each of the grid points \( x_i \), there exists a backward characteristic line with the foot located on the time level \( t^n \) at \( y_i = x_i - c\Delta t \) for linear advection equation with advection speed \( c \). Along the characteristics, the solution is constant \( u(x_i, t^{n+1}) = u(x_i - c\Delta t, t^n) \). If we directly apply some point value reconstruction, e.g. the WENO interpolation to update \( u(x_i, t^{n+1}) \), the numerical scheme is not necessarily conservative. To design a conservative SL FD scheme, we define an \( h(x, t) \) function as following,

\[
\frac{1}{\Delta x} \int_{X(t;x - \frac{h}{2},t^n)}^{X(t;x + \frac{h}{2},t^{n+1})} h(\xi, t)d\xi = u(x, t), \quad t \in [t^n, t^{n+1}],
\]

(3.3)

where \( X(t;\xi,t^{n+1}) \) are the characteristics curves over \([t^n, t^{n+1}]\) ending at \((\xi, t^{n+1})\), i.e.

\[
\frac{dX(t)}{dt} = c, \quad X(t^{n+1}) = \xi.
\]

From the equation (3.3), we know that

\[
u_i^{n+1} = \frac{1}{\Delta x} \int_{x_i - \frac{1}{2}}^{x_i + \frac{1}{2}} h(\xi, t^{n+1})d\xi, \quad \frac{1}{\Delta x} \int_{x_i - \frac{1}{2} - c\Delta t}^{x_i + \frac{1}{2} - c\Delta t} h(\xi, t^n)d\xi = u(x_i - c\Delta t, t^n).
\]

Since \( u(x_i, t^{n+1}) = u(x_i - c\Delta t, t^n) \), we have

\[
\frac{1}{\Delta x} \int_{x_i - \frac{1}{2}}^{x_i + \frac{1}{2}} h(\xi, t^{n+1})d\xi = \frac{1}{\Delta x} \int_{x_i - \frac{1}{2} - c\Delta t}^{x_i + \frac{1}{2} - c\Delta t} h(\xi, t^n)d\xi.
\]

(3.4)
In other words, in order to update the point values of $u$, e.g. $u_{j}^{n+1}$ from $u(x_{i} - c\Delta t, t^{n})$, it is equivalent to update the cell averages of $h$, e.g. \( \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} h(\xi, t^{n+1})d\xi \) from some integrated mass of $h$ at $t^{n}$ on the R.H.S. of equation (3.4). On the other hand, this integral can be reconstructed from the cell averages of $h$, since

\[
\bar{h}_{i} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} h(\xi, t^{n})d\xi \equiv u_{i}^{n},
\]

in the same way as the WENO reconstruction in the SL FV WENO scheme described in the previous subsection.

**Remark 3.1.** (conservation) The conservation property of the scheme can be seen by

\[
\Delta x \sum_{i=1}^{N} u_{i}^{n+1} = \Delta x \sum_{i=1}^{N} \bar{h}_{i}^{n+1} = \sum_{i=1}^{N} \int_{x_{i-1/2}}^{x_{i+1/2}} h(\xi, t^{n})d\xi = \Delta x \sum_{i=1}^{N} \bar{h}_{i}^{n} = \Delta x \sum_{i=1}^{n} u_{i}^{n},
\]

if periodic boundary condition is assumed.

**Remark 3.2.** Unfortunately, the above formulation cannot be applied to the equations with variable coefficients. The problem is that

\[
u_{i}^{n} = \frac{1}{\Delta x} \int_{X(t^{n}; x_{i + 1/2}, t^{n+1})}^{X(t^{n}; x_{i - 1/2}, t^{n+1})} h(\xi, t)d\xi \neq \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} h(\xi, t^{n})d\xi = \bar{h}_{i}^{n}.
\]

In other words, $u_{i}^{n}$ is not necessarily the cell averages of $h$, $\bar{h}_{i}^{n}$. A conservative update via this route is therefore, if not impossible, highly non-trivial.

**Remark 3.3.** The SL FV/FD formulations for the linear advection equations with constant coefficients are equivalent to the Lax-Wendroff formulation with enough terms in the Taylor expansions. This is a different time discretization strategy from the SSP RK schemes.

## 4 Numerical tests

In this section, the SL FD schemes proposed in Section 3 (referred to as Method I) is tested for the simple cases of linear advection and rigid body rotation, together with the conservative SL FD schemes proposed in [21, 20] (referred as Method II and Method III respectively), and the non-conservative SL FD scheme proposed in [3] (referred as Method IV). All of the SL FD schemes are coupled with a fifth order WENO reconstruction/interpolation.
Table 1: Order of accuracy for (4.1) with \(u(x, t = 0) = \sin(x)\) at \(T = 20\). \(CFL = 2.2\).

<table>
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<th>Method I order</th>
<th>Method II error</th>
<th>Method II order</th>
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<td>5.03</td>
<td>1.12E-8</td>
<td>5.02</td>
<td>1.24E-8</td>
<td>5.02</td>
<td>1.09E-8</td>
<td>5.02</td>
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<tr>
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<td>3.67E-9</td>
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<td>4.99</td>
<td>3.56E-9</td>
<td>5.04</td>
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</table>

4.1 Test examples

Example 4.1. (one dimensional linear translation)

\[ u_t + u_x = 0, \quad x \in [0, 2\pi]. \tag{4.1} \]

Four different SL FD methods (Methods I, II, III, IV) with fifth order WENO reconstruction/interpolation are used to solve equation (4.1). Table 1 gives the \(L^1\) error, and the corresponding order of convergence when the four different methods are applied to equation (4.1) with the smooth initial data \(u(x, 0) = \sin(x)\). As expected, fifth order convergence is observed. The schemes also inherit the essentially non-oscillatory property of the WENO reconstruction, when advecting rectangular waves. Numerical results are omitted here to save space. For Method IV, the conservation error is not significant for this example. The conservation error up to \(T = 20\) is in the order of \(10^{-15}\) for the smooth sine wave function, and is in the order of \(10^{-12}\) for the rectangular wave. The non-conservative scheme seems to have slightly smaller \(L^1\) error in magnitude.

Example 4.2. (two dimensional linear transport)

\[ u_t + u_x + u_y = 0, \quad x \in [0, 2\pi], \quad y \in [0, 2\pi]. \tag{4.2} \]

The equation is being split into two one-dimensional equations, each of which is evolved by SL FD WENO methods. For any 2-D linear transport equation, the SL method is essentially a shifting procedure. Since the \(x\)-shifting and \(y\)-shifting operators commute, there is no dimensional splitting error in time and the spatial error is the dominant error. Table 2 gives the \(L^1\) error and the corresponding order of convergence for applying the four different SL
Table 2: Order of accuracy for (4.2) with \( u(x,y,t = 0) = \sin(x + y) \) at \( T = 20 \). CFL = 2.2.

<table>
<thead>
<tr>
<th>mesh</th>
<th>Method I error order</th>
<th>Method II error order</th>
<th>Method III error order</th>
<th>Method IV error order</th>
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<td>20x20</td>
<td>7.94E-4 -</td>
<td>7.94E-4 -</td>
<td>8.28E-4 -</td>
<td>6.03E-4 -</td>
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<td>7.80E-7 5.00</td>
<td>8.16E-7 5.00</td>
<td>7.50E-7 4.93</td>
</tr>
</tbody>
</table>

FD schemes to equation (4.2) with the smooth solution \( u(x,y,t) = \sin(x + y - 2t) \). Again fifth order convergence for all schemes are observed as expected. In our 2-D simulation, \( CFL = \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta y} \).

**Example 4.3.** (rigid body rotation)

\[
u_t - yu_x + xu_y = 0, \quad x \in [-2\pi, 2\pi], \quad y \in [-2\pi, 2\pi].
\]  

(4.3)

The equation is being Strang split into two one-dimensional equations, each of which is evolved by different SL FD WENO methods. The initial condition we used is plotted in Figure 1. It includes a slotted disk, a cone as well as a smooth hump, similar to the one in [18] for comparison purpose. The numerical solutions after six full revolutions by the schemes are plotted in Figure 2 by 2D surfaces and in Figure 3 by 1D cuts benchmarked with the exact solution. With all the reconstructions, non-oscillatory capturing of discontinuities is observed. Conservative schemes (Methods I, II, III) are observed to perform better than the non-conservative SL FD WENO scheme (Method IV), see the first and second plots in Figure 3.

## 5 The Vlasov-Poisson system

In this section, we demonstrate the performance of the proposed method, compared with those in [20, 21, 3] by applying them to classical problems in plasma physics, such as Landau damping and two-stream instability. These classical phenomena are described by the well-known Vlasov-Poisson (VP) system

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + E(t,x) \cdot \nabla_v f = 0,
\]

(5.4)
Figure 1: Plots of the initial profile. The numerical mesh is $100 \times 100$.

$$E(t, x) = -\nabla_x \phi(t, x), \quad -\Delta_x \phi(t, x) = \rho(t, x). \quad (5.5)$$

In the equations (5.4) - (5.5), $x$ and $v$ are the coordinates in the phase space $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$, $E$ is the electric field, $\phi$ is the self-consistent electrostatic potential and $f(t, x, v)$ is the probability distribution function which describes the probability of finding a particle with velocity $v$ at position $x$ at time $t$. The probability distribution function couples to the long range fields via the charge density, $\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v) dv - 1$, where we take the limit of uniformly distributed infinitely massive ions in the background. Equations (5.4) and (5.5) have been nondimensionalized so that all physical constants are one. Below, we briefly recall some classical preservation results in the VP system. We hope that our numerical solutions can preserve these classical conserved quantities as much as possible.

1. Preservation of the $L^p$ norm, for $1 \leq p < \infty$.

$$\frac{d}{dt} \int_v \int_x f(x, v, t)^p dx dv = 0 \quad (5.6)$$

2. Preservation of the entropy

$$\frac{d}{dt} \int_v \int_x f(x, v, t) \log(f(x, v, t)) dx dv = 0 \quad (5.7)$$
3. Preservation of the energy

$$\frac{d}{dt} \left( \int_v \int_x f(x, v, t)v^2 dv + \int_x E^2(x, t) dx \right) = 0$$  \hspace{1em} (5.8)

The Strang splitting SL method for the VP system was originally proposed in reference [4], and soon gained wide popularity [10, 27, 9]. The Strang splitting reduces the high dimensional nonlinear Vlasov equation into one-dimensional advection equations, on which the high order SL FD WENO schemes can be applied. In this section, the four different SL formulations tested in Section 4 are applied to the Strang split Vlasov equation. The schemes are tested on the classical problems in plasma physics, such as Landau damping and
two stream instabilities. The performance of the schemes will be demonstrated/compared by
the solution profiles, as well as by tracking the time evolution of the theoretically preserved
quantities (equation (5.6) - (5.8)) in the discrete sense.

Without loss of generality, we consider the Vlasov equation, equation (5.4), with only
one position and one velocity axis, i.e., \((x, v) \in \mathbb{R} \times \mathbb{R}\). The extension to higher dimensions
in \(x\) and \(v\) is straightforward. The time splitting form of equation (5.4) is,

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0 , \tag{5.9}
\]

\[
\frac{\partial f}{\partial t} + E(t, x) \frac{\partial f}{\partial v} = 0 . \tag{5.10}
\]

The split form of equation (5.4) can be made second order accurate in time by solving
equation (5.9) for a half time step, then solving equation (5.10) for a full time step, followed
by solving equation (5.9) for a second half time step. The observation that both equation
(5.9) and equation (5.10) are linear hyperbolic equations allows for a direct implementation
of the SL FD WENO schemes. Specifically, the numerical update from \(f^n(x, v)\) (the solution
at \(t^n = n\Delta t\)) to \(f^{n+1}(x, v)\) is as follows:

1. Advance a half time step for equation (5.9) by a SL method,

\[f^*(x, v) = SL(f, \frac{1}{2}\Delta t); \tag{5.11}\]

2. Compute the electric field at the half step by substituting \(f^*\) into equation (5.5) and
solve for \(E^*(x)\);

3. Advance a full time step for equation (5.10) by a SL method,

\[f^{**}(x, v) = SL(f^*, \Delta t); \tag{5.12}\]

4. Advance a half time step for equation (5.9) by a SL method,

\[f^{n+1}(x, v) = SL(f^{**}, \frac{1}{2}\Delta t). \tag{5.13}\]

In our experiments, periodic boundary conditions are imposed in the \(x\)-direction and zero
boundary conditions are imposed in the \(v\)-direction for all of our test problems. Because of
the periodicity in space, a fast Fourier transform (FFT) is used to solve the 1-D Poisson equation. \( \rho(x,t) \) is computed by the rectangular rule, \( \rho(x,t) = \int f(x,v,t)dv = \sum_j f(x,v_j,t)\Delta v \), which is spectrally accurate [1], when the underlying function is smooth enough. In our numerical experiments below, the \( L^p \) norms/entropy/energy are numerically approximated by the rectangular rule, which is again spectrally accurate, if the integrated function is smooth enough.

**Example 5.1.** (Weak Landau damping) Consider the example of weak Landau damping for the VP system. The initial condition used here is,

\[
f(x,v,t=0) = \frac{1}{\sqrt{2\pi}}(1 + \alpha \cos(kx)) \exp\left(-\frac{v^2}{2}\right),
\]

with \( \alpha = 0.01 \) and \( k = 0.5 \). The time evolution of the \( L^2 \) and \( L^\infty \) norms of the electric field is plotted in the upper plots of Figure 4. The correct damping of the electric field is observed in the plots, benchmarked with the theoretical value \( \gamma = 0.1533 \) [11] (the solid line in the same plots). We observe that all of the four methods generate very consistent results, performing very well in recovering the damping rate. The time evolution of the \( L^1 \), \( L^2 \) solution norms, energy, entropy in the discrete sense are demonstrated in the middle and bottom plots in Figure 4. The advantage of using conservative schemes in preserving the relevant physical norms is observed. Despite this, we remark that in the weak Landau damping case, the relevant physical norms are preserved pretty well for both the conservative and non-conservative schemes (see the magnitude variance in all of the \( y \)-axis) in Figure 4.

**Example 5.2.** (Strong Landau damping) The next example we consider is the case of strong Landau damping. We simulate the VP system with the initial condition in equation (5.14) with \( \alpha = 0.5 \) and \( k = 0.5 \). Our numerical simulation parameters for all schemes are \( v_{\text{max}} = 5 \), \( N_x = 64 \), \( N_v = 128 \) and \( \Delta t = \Delta x \); where \( v_{\text{max}} \) is the maximum velocity on the phase space mesh, \( N_x \) is the number of grid points along the \( x \) axis, \( N_v \) is the number of grid points along the \( v \) axis, and \( \Delta t \) is the time step used. In the first row of Figure 5, the time evolution of the \( L^2 \) and \( L^\infty \) norms of the electric field is plotted. The profile of the non-conservative method IV deviates from the evolution profiles of other conservative SL FD WENO methods after a long time evolution (roughly around \( T=40 \)). The discrete \( L^1 \) norm, \( L^2 \) norm, kinetic energy and entropy for four different methods are plotted in the second and third rows.
of Figure 5. It is observed that schemes with conservative properties do a better job in preserving the discrete $L^1$, $L^2$ norms and kinetic energy than a non-conservative scheme. On the other hand, for some unknown reason, the non-conservative scheme (method IV) seems to be better in preserving the entropy for this test case. Since the numerical solutions we obtained are consistent with those in Figure 4.9 in [20], we skip demonstrating them to save space.

**Example 5.3.** (Two stream instability [11]) Consider the symmetric warm two stream instability, i.e., the electron distribution function in the VP system is started with the unstable initial condition [11],

$$f(x, v, t = 0) = \frac{2}{7\sqrt{2\pi}} (1 + 5v^2)(1 + \alpha((\cos(2kx) + \cos(3kx))/1.2 + \cos(kx)) \exp\left(-\frac{v^2}{2}\right),$$

(5.15)

with $\alpha = 0.01$, $k = 0.5$. The length of the domain in the $x$ direction is $L = \frac{2\pi}{k}$ and the background ion distribution function is fixed, uniform and chosen so that the total net charge density for the system is zero. Our numerical simulation parameters are $v_{\text{max}} = 5$, $N_x = 64$, $N_v = 128$, $\Delta t = \Delta x$ for all the schemes. Figure 6 shows numerical solutions of phase space profiles at $T = 53$ from the four different SL FD WENO schemes. The conservative schemes (methods I, II, III) seem to perform slightly better than the non-conservative scheme, if careful observation in the rotational core is made. In the first row of Figure 7, the time evolution of the $L^2$ and $L^\infty$ norms of the electric field is plotted. Consistent numerical solutions from all of the four methods are observed. The second and third rows of Figure 7 are the time development of the discrete $L^1$ norm, $L^2$ norm, kinetic energy and entropy for the four different methods. It is observed that schemes with conservative properties in general do a better job in preserving those physical norms than a non-conservative scheme.

**Example 5.4.** (Two stream instability [9]) Consider the symmetric two stream instability, similar as in [9],

$$f(x, v, t = 0) = \frac{1}{2v_{\text{th}}\sqrt{2\pi}} \left[\exp\left(-\frac{(v - u)^2}{2v_{\text{th}}^2}\right) + \exp\left(-\frac{v + u}{2v_{\text{th}}^2}\right)\right] (1 + 0.05 \cos(kx))$$

(5.16)

with $u = 0.99$, $v_{\text{th}} = 0.3$ and $k = \frac{2\pi}{13}$. The background ion distribution function is fixed, uniform and chosen so that the total net charge density for the system is zero. Our numerical
simulation parameters are \( v_{\text{max}} = 5, N_x = 512, N_v = 512, \Delta t = \Delta x \) for all the schemes. Figure 8 shows numerical solutions of phase space profiles at \( T = 70 \) from the four different SL FD WENO schemes. The conservative schemes (methods I, II, III) seem to perform better than the non-conservative scheme, compared with the double refined reference solution shown in the last plot in Figure 8. In the first row of Figure 9, the time evolution of the \( L^2 \) and \( L^\infty \) norms of the electric field is plotted. Consistent numerical solutions from all of the four methods are observed. The second and third rows of Figure 9 are the time development of the discrete \( L^1 \) norm, \( L^2 \) norm, kinetic energy and entropy for the four different methods. Again, conservative schemes in general perform better in preserving those physical norms than the non-conservative scheme.

6 Conclusions

In this paper, we propose a new conservation semi-Lagrangian (SL) finite difference (FD) WENO scheme, based on the same reconstruction procedure as the one used in a SL finite volume (FV) WENO scheme. We implement the proposed scheme, as well as the other three SL FD WENO schemes in [3, 20, 21] to the linear advection, rigid body rotation problem; and on the Landau damping and two-stream instabilities by solving the VP system. We compare the performance of different schemes, and demonstrate that conservative schemes in general perform better than non-conservative ones in tracking the evolution of physically conserved quantities.

References


Figure 3: Plots of the 1-D cuts of the numerical solution for equation (4.3) at \( X = 0 \) (top), \( Y = -1.6 \) (middle) and \( Y = 1.54 \) (bottom) with CFL = 2.2 at \( T = 12\pi \). The numerical mesh is 100 × 100.
Figure 4: Weak Landau damping: time evolution of the electric field in $L^2$ (upper left) and $L^\infty$ (upper right) norms, $L^1$ (middle left) and $L^2$ (middle right) norms of the solution as well as the discrete kinetic energy (lower left) and entropy (lower right).
Figure 5: Strong Landau damping: time evolution of the electric field in $L^2$ (upper left) and $L^\infty$ (upper right) norms, $L^1$ (middle left) and $L^2$ (middle right) norms of the solution as well as the discrete kinetic energy (lower left) and entropy (lower right).
Figure 6: Phase space plots of the two stream instability at $T = 53$ using methods I, II, III, IV. The numerical mesh is $64 \times 128$. 
Figure 7: Two-stream instability: time evolution of the electric field in $L^2$ (upper left) and $L^\infty$ (upper right) norms, $L^1$ (middle left) and $L^2$ (middle right) norms of the solution as well as the discrete kinetic energy (lower left) and entropy (lower right).
Figure 8: Phase space plots of the two stream instability at $T = 70$ using methods I, II, III, IV. The numerical mesh is $512 \times 512$. The bottom plot is the same solution but using double refined numerical mesh $1024 \times 1024$ as a reference solution.
Figure 9: Two-stream instability: time evolution of the electric field in $L^2$ (upper left) and $L^\infty$ (upper right) norms, $L^1$ (middle left) and $L^2$ (middle right) norms of the solution as well as the discrete kinetic energy (lower left) and entropy (lower right).