1 Physical models and numerical test

1.1 Physical models

Problem 4.1: A sphere of non-zero opacity matter

Equations about $J'(t,r)$ is the following:

\[
\frac{\partial J'}{\partial ct} + \frac{\partial J'}{\partial r} = -kJ', r \neq 0
\]  

(1.1)

with the boundary condition:

\[
J'(t, r = 0) = \frac{E}{4\pi}
\]  

(1.2)

and initial condition:

\[
J'(t = 0, r) = \begin{cases} 
0, & \text{if } r > 0 \\
\frac{E}{4\pi}, & \text{if } r = 0 
\end{cases}
\]  

(1.3)

The exact solution for the equation (1.1) with the corresponding boundary condition and initial condition is the following:

\[
J'(t, r) = \theta(ct - r) \frac{E}{4\pi} e^{-kr}
\]  

(1.4)

$\theta(x)$ is the step function, taking value in 1 for $x \geq 0$ and 0 otherwise.

Eq.(0.1) can be rewritten as

\[
\frac{\partial J'}{\partial kct} + \frac{\partial J'}{\partial kr} = -J', r \neq 0
\]  

(1.5)

Let $t' = kct$ and $r' = kr$, then we have

\[
\frac{\partial J'}{\partial t'} + \frac{\partial J'}{\partial r'} = -J', r \neq 0,
\]  

(1.6)

where the variables $r'$ and $t'$ are dimensionless. We can take $E = 1$. The exact solution for the equation (1.6) with the corresponding boundary condition and initial condition is the
following:

\[ J'(t,r) = \theta(t - r) \frac{E}{4\pi} e^{-r} \]  

(1.7)

**Problem 4.2: Stromgren sphere**

Equation about \( J'(t,r) \) is the following:

\[ \frac{\partial J'}{\partial ct} + \frac{\partial J'}{\partial r} = -\sigma(\nu_0)n_{HI}(t,r)J', r \neq 0 \]  

(1.8)

with the boundary condition:

\[ J'(t, r = 0) = \frac{E}{4\pi} \]  

(1.9)

and the initial condition:

\[ J'(t = 0, r) = \begin{cases} 0, & \text{if } r > 0 \\ \frac{E}{4\pi}, & \text{if } r = 0 \end{cases} \]  

(1.10)

\( \sigma(\nu_0) \) in (1.8) = 16.3 \times 10^{-18} \text{ cm}^2

and \( n_{HI}(t, r) \) determined by the following quadratic equation:

\[ (\alpha_{HI} + \Gamma_{eHI})n_{HI}^2 - (\Gamma_{\gamma HI} + 2n\alpha_{HI} + n\Gamma_{eHI})n_{HI} + n^2\alpha_{HI} = 0 \]

Since the \( \Gamma_{eHI} \) is much smaller than \( \alpha_{HI} \), one ignores \( \Gamma_{eHI} \) in the equation above:

\[ (\alpha_{HI})n_{HI}^2 - (\Gamma_{\gamma HI} + 2n\alpha_{HI})n_{HI} + n^2\alpha_{HI} = 0 \]  

(1.11)

where

\[ \alpha_{HI} = \frac{6.30 \times 10^{-11} T \frac{2}{5} T_3^{-0.3}}{1 + T_6^{0.7}} = 1.97653475299427 E - 012 \]

\[ \Gamma_{eHI} = 1.17 \times 10^{-10} T^{\frac{7}{5}} e^{-157.809.1/T} (1 + T_5^{1.3})^{-1} = 9.798776772381929 E - 078 \]

\( T = 10^3; T_3 = T/10^3 \); \( T_5 = T/10^5 \); \( T_6 = T/10^6 \); \( e = 2.718281828 \), and \( \Gamma_{\gamma HI} = \frac{1}{r^2} \frac{J'(t,r)}{h\nu_0} \sigma_0 \)
Let \( t' = \sigma(\nu_0) n c t, \quad r' = \sigma(\nu_0) n r, \quad f_{HI} = \frac{\alpha_{HI}}{n} \) and \( J'' = J' \frac{\sigma_0}{\nu_0} \), equation (1.8) can be rewritten as:

\[
\frac{\partial J''}{\partial t'} + \frac{\partial J''}{\partial r'} = - f_{HI} J', \quad r \neq 0
\]

(1.12)

Here \( f_{HI} = \frac{\alpha_{HI}}{n} \) (0 \( \leq f_{HI} \leq 1 \)) is determined by:

\[
\alpha_{HI} J''_{HI}^2 - (\Gamma_{\gamma HI}/n + 2\alpha_{HI}) f_{HI} + \alpha_{HI} = 0
\]

(1.13)

where \( \Gamma_{\gamma HI}' = \frac{\Gamma_{\gamma HI}}{n} = \frac{J''}{\pi r} \). The boundary and initial condition for the rescale differential equation is:

\[
J''(t, r = 0) = \frac{E}{4\pi}
\]

(1.14)

and the initial condition:

\[
J''(t = 0, r) = \begin{cases} 
0, & \text{if } r > 0 \\
\frac{E}{4\pi}, & \text{if } r = 0 
\end{cases}
\]

(1.15)

Observe that in equation (1.13), the coefficient for the second term consist of \( \Gamma_{\gamma HI}' = \frac{J''}{\pi r} \)
and \( \alpha_{HI} \), which is on the order of \( 10^{-12} \).

If the \( E \) in the initial condition and boundary condition is 1.0/ 0.1/ 0.01, then the root of the quadratic equation will be very close to 0.

If the \( E \) in the initial condition and boundary condition is in the same order as \( \alpha \), then the root of the quadratic equation \( f_{HI} \) will be distributed more evenly in the interval.

Therefore, I rescaled \( \alpha = 10^{12} \alpha \) (s.t. it is of order 1) and \( J'' = 10^{12} J'' \). All the equations above stay the same, except that the number is being amplified \( 10^{12} \) times.

**Problem 4.3: Stromgren sphere with non-monochromatic source**

Equation about \( J'(t, r, \nu) \) is the following:

\[
\frac{\partial J'}{\partial t} + \frac{\partial J'}{\partial r} = - k_\nu J', \quad r \neq 0
\]

(1.16)
with the boundary condition:

\[ J'(t, r = 0) = \frac{E(\nu)}{4\pi} \tag{1.17} \]

and the initial condition:

\[ J'(t = 0, r) = \begin{cases} 0, & \text{if } r > 0 \\ \frac{E(\nu)}{4\pi}, & \text{if } r = 0 \end{cases} \tag{1.18} \]

where \( E(\nu) \) be a function of \( \nu \)

Take \( \nu_0 \) be the unit of \( \nu \), \( k_\nu = \sigma(\nu)n_{HI}(t, r) = \sigma(\nu_0)\frac{1}{\nu_0^2}n_{HI}(t, r) \)

with \( \sigma(\nu) = 6.3 \times 10^{-18}(\frac{\nu}{\nu_0})^3 \text{cm}^2 \)

and \( n_{HI}(t,r) \) be the solution of quadratic equations as in problem 4.2, except that

\[ \Gamma_{\gamma HI}(t, r) = \frac{1}{r^2} \int_{\nu_0}^{\infty} d\nu \frac{J'(t, r, \nu)}{h\nu} \sigma(\nu) \]

Let \( \ell = c\nu\sigma(\nu_0)t, r' = n\sigma(\nu_0)r, \nu' = \frac{\nu}{\nu_0} \) and \( J'' = J'\frac{n\sigma(\nu_0)^3}{h} \)

Then the updated equation is the following:

\[ \frac{\partial J''}{\partial \ell'} + \frac{\partial J''}{\partial r'} = -(\frac{1}{\nu'})^3 f_{HI} J'' \]

where \( f_{HI} \) be the solution of the quadratic equation:

\[ (\alpha_{HI})f_{HI}^2 - (\Gamma_{\gamma HI} + 2\alpha_{HI})f_{HI} + \alpha_{HI} = 0 \]

notice that

1. \( \Gamma_{eHI} \) is ignored, since it is much smaller when compared with \( \alpha_{HI} \)
2. \( \Gamma_{\gamma HI} = \frac{1}{r^2} \int_{1}^{\infty} \frac{J''}{\nu^4} d\nu \quad (1.19) \)
The boundary and initial conditions can be

\[ J''(t, r = 0, \nu) = \frac{E}{4\pi \nu} \frac{1}{r} \]  

(1.20)

and the initial condition:

\[ J''(t = 0, r, \nu) = \begin{cases} 
0, & \text{if } r > 0 \\
\frac{E}{4\pi \nu}, & \text{if } r = 0
\end{cases} \]  

(1.21)

Similar as in Problem 4.2, the \( \alpha \) and \( J'' \) is amplified to \( 10^{12} \) times in computation.

1.2 Numerical scheme

The equations derived above are solved by the fifth order conservative finite difference WENO scheme with anti-diffusive flux corrections coupled with a third order Runge-Kutta time discretization.

Take Problem 4.3 for example, the computational domain is discretized into a tensor product mesh. The uniform mesh is taken in \( r \)-direction, a smooth non-uniform mesh is taken in the \( \nu \)-direction:

\[ r_i = i\Delta r; \quad i = 0, 1, 2, ..., N_r, \]
\[ \nu_j = 2^{\xi_j}; \quad j = 0, 1, 2, ..., N_\nu, \]  

(1.22)

where \( \Delta r = r_{\text{max}}/N_r \) is the mesh size in \( r \)-direction. And \( \xi_j = j\Delta \xi, \Delta \xi = \log_2\nu_{\text{max}}/N_\nu \) is the transformed mesh size in \( \nu \) direction. \( r_{\text{max}} \) and \( \nu_{\text{max}} \) are the lengths of the numerical domain, which are adjusted in the numerical experiment such that \( J(t, r, \nu) \approx 0 \), for \( r > r_{\text{max}} \) for every \( t \) and \( \nu \) and \( J(t, r, \nu) \approx 0 \), for \( \nu > \nu_{\text{max}} \) for every \( t \) and \( r \). The approximations to the point values of the solution \( J(t^n, r_i, \nu_j) \), denoted by \( J^n_{i,j} \), are obtained with an approximation to the spacial dirivatives using 5th order WENO method with anti-diffusive flux corrections and the approximation to the source term.

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• approximation to the derivatives:

To approximate $\frac{\partial J}{\partial r}$, the variable $\nu$ is fixed and the approximation is performed along the r-line:

$$\frac{\partial}{\partial r} J(t^n, r_i, \nu_j) \approx \frac{1}{\Delta r} \left( \hat{h}_{i+1/2}^q - \hat{h}_{i-1/2}^q \right)$$

where the numerical flux $\hat{h}_{i+1/2}^q$ is obtained with the following procedure. Notice that the wind direction is simply from left to right, we use the upwind fluxes in the fifth order WENO approximation. To obtain the sharp resolution at the contact discontinuities, the anti-diffusive flux corrections are used in the scheme. We denote

$$h_i = J(t^n, r_i, \nu_j), i = -2, -1, ..., N_r + 2$$

where $n$ and $j$ are fixed.

The numerical flux from the regular WENO procedure is obtained by

$$\hat{h}_{i+1/2} = \omega_1 \hat{h}_{i+1/2}^{(1)} + \omega_2 \hat{h}_{i+1/2}^{(2)} + \omega_3 \hat{h}_{i+1/2}^{(3)}$$

where $\hat{h}_{i+1/2}^{(m)}$ are the three third order fluxes on three different stencils given by

$$\hat{h}_{i+1/2}^{(1)} = \frac{1}{3} h_{i-2} - \frac{7}{6} h_{i-1} + \frac{11}{6} h_i,$$
$$\hat{h}_{i+1/2}^{(2)} = -\frac{1}{6} h_{i-1} + \frac{5}{6} h_i + \frac{1}{3} h_{i+1},$$
$$\hat{h}_{i+1/2}^{(3)} = \frac{1}{3} h_i + \frac{5}{6} h_{i+1} - \frac{1}{6} h_{i+2},$$

and the nonlinear weights $\omega_m$ are given by

$$\omega_m = \frac{\tilde{\omega}_m}{\sum_{l=1}^{3} \tilde{\omega}_l}, \quad \tilde{\omega}_l = \frac{\gamma_l}{(\varepsilon + \beta_l)^2},$$

with the linear weights $\gamma_l$ given by

$$\gamma_1 = \frac{1}{10}, \quad \gamma_2 = \frac{3}{5}, \quad \gamma_3 = \frac{3}{10},$$
and the smoothness indicators $\beta_i$ given by

\begin{align*}
\beta_1 &= \frac{13}{12} (h_{i-2} - 2h_{i-1} + h_i)^2 + \frac{1}{4} (h_{i-2} - 4h_{i-1} + 3h_i)^2 \\
\beta_2 &= \frac{13}{12} (h_{i-1} - 2h_i + h_{i+1})^2 + \frac{1}{4} (h_{i-1} - h_{i+1})^2 \\
\beta_3 &= \frac{13}{12} (h_i - 2h_{i+1} + h_{i+2})^2 + \frac{1}{4} (3h_i - 4h_{i+1} + h_{i+2})^2.
\end{align*}

$\varepsilon$ is a parameter to avoid the denominator to become 0 and is taken as $10^{-6} \times M$, with $M$ be the magnitude of the initial condition, in the computation of this paper.

The anti-diffusive flux corrections are based on the fluxes obtained from the regular WENO procedure.

\[
\hat{h}_{i+1/2}^g = \hat{h}_{i+1/2} + \phi_{i+1/2} \text{minmod} \left( \frac{\hat{h}_i - \hat{h}_{i-1}}{\lambda}, \hat{h}_{i-1/2} - \hat{h}_{i+1/2}, \hat{h}_{i+3/2} - \hat{h}_{i+1/2} \right),
\]

where $\lambda = \Delta t / \Delta x$ is the cfl number and the minmod function is defined as,

\[
\text{minmod}(a, b) = \begin{cases} 
0, & \text{if } ab \leq 0 \\
abla, & \text{if } ab > 0, |a| \leq |b| \\
\sigma, & \text{if } ab > 0, |b| \leq |a|
\end{cases}
\]

where $\phi_{i+1/2}$ is the discontinuity indicator between 0 and 1. It is defined as:

\[
\phi_i = \frac{\beta_i}{\beta_i + \gamma_i},
\]

where

\begin{align*}
\alpha_i &= |h_{i-1} - h_i|^2 + \varepsilon, \\
\beta_i &= \left( \frac{\alpha_i}{\alpha_{i-1}} + \frac{\alpha_{i+1}}{\alpha_{i+2}} \right)^2, \\
\gamma_i &= \frac{|u_{\text{max}} - u_{\text{min}}|^2}{\alpha_i},
\end{align*}

where $\varepsilon$ is a small positive number taken as $10^{-6}$ in computation, and $u_{\text{max}}$ and $u_{\text{min}}$ are the maximum and minimum value of $h_i$ for all grid points. With the definition
above, we will have $0 \leq \phi_i \leq 1$. $\phi_i = O(\Delta x^2)$ in the smooth regions and $\phi_i$ is close to 1 near strong discontinuities.

- Boundary conditions
  We only need to take care of the boundary condition for $r$-direction, as no derivative is involved in the $\nu$ direction.
  At $r=0$, we use the inflow boundary condition:
  \[ J_{1-i,j} = E(\nu)/4\pi, \quad \text{for} \quad i = 1, 2, 3 \]
  At $r=r_{max}$, the boundary condition is:
  \[ J_{Nz+i,j} = J_{Nz-1,j}, \quad \text{for} \quad i = 0, 1, 2 \]

- Integration
  Numerical integration, Eq(1.19), is involved in obtaining the coefficient $f_{HI}$ in the source term.
  A quadrature formula accurate of order 4 is used in computation:
  \[
  \int_0^\infty f(x)dx = dx \sum_{j=0}^\infty w_j f(jdx) + O(dx^4) \tag{1.24}
  \]
  where the weight $w_j$ is the following,
  \[ w_0 = \frac{3}{8}, w_1 = \frac{7}{6}, w_2 = \frac{23}{24}, w_j = 1 \quad \text{for} \quad j > 2 \]
  By having a non-uniform mesh in $\nu$-direction, one can’t use this quadrature formula directly. Notice that a uniform mesh size is used on $\xi$, with $\nu_j = 2^\xi_j$, we can do the numerical integration with respect to $\xi$.

- Third order Runge-Kutta discretization
The time discretization is by the following third order TVD Runge-Kutta method:

\[
\begin{align*}
J^{(1)} & = J^n + \Delta t L(J^n, t^n) \\
J^{(2)} & = J^n + \frac{1}{4} \Delta t L(J^n) + \frac{1}{4} \Delta t L(J^{(1)}) \\
J^{n+1} & = J^n + \frac{1}{6} \Delta t L(J^n) + \frac{1}{6} \Delta t L(J^{(1)}) + \frac{2}{3} \Delta t L(J^{(2)})
\end{align*}
\]

where \( L \) is the approximation to the spatial derivatives and the source terms:

\[
L(J) \approx -\frac{\partial}{\partial x} J - \left( \frac{1}{\nu} \right)^3 f_{HI} J
\]

The Runge-Kutta method needs to be modified with the modification on the anti-diffusive flux \( \tilde{f}^a \) by

\[
\begin{align*}
J^{(1)} & = J^n + \Delta t L(J^n, t^n) \\
J^{(2)} & = J^n + \frac{1}{4} \Delta t L(J^n) + \frac{1}{4} \Delta t L(J^{(1)}) \\
J^{n+1} & = J^n + \frac{1}{6} \Delta t L(J^n) + \frac{1}{6} \Delta t L(J^{(1)}) + \frac{2}{3} \Delta t L(J^{(2)})
\end{align*}
\]

where the operator \( L \) is defined by the anti-diffusive flux \( \hat{h}^a \), the operator \( L' \) is defined by the modified anti-diffusive flux \( \overline{h}^a \)

\[
\overline{h}^a_{i+1/2} = \begin{cases} 
\hat{h}^a_{i+1/2} + \text{minmod} \left( \frac{4(h_i - h_{i-1})}{\lambda}, \hat{h}_{i-1/2} - \hat{h}_{i+1/2}, \hat{h}_{i+1/2} - \hat{h}_{i+3/2} \right), & i \text{f} \quad bc > 0, |b| < |c|, \\
\hat{h}^a_{i+1/2}, & \text{otherwise}
\end{cases}
\]

and \( L'' \) is defined by the modified anti-diffusive flux \( \tilde{h}^a \),

\[
\tilde{h}^a_{i+1/2} = \begin{cases} 
\hat{h}^a_{i+1/2} + \text{minmod} \left( \frac{6(h_i - h_{i-1})}{\lambda}, \hat{h}_{i-1/2} - \hat{h}_{i+1/2}, \hat{h}_{i+1/2} - \hat{h}_{i+3/2} \right), & i \text{f} \quad bc > 0, |b| < |c|, \\
\hat{h}^a_{i+1/2}, & \text{otherwise}
\end{cases}
\]

in equations (1.25)(1.26) \( b = (h_i - h_{i-1})/\lambda + \hat{h}_{i-1/2} - \hat{h}_{i+1/2}, \quad c = \hat{h}_{i+3/2} - \hat{h}_{i+1/2} \).

### 1.3 Numerical result

In this section, we will show the performance of the fifth order finite difference WENO scheme with the anti-diffusive flux corrections to the equations.
**Problem 4.1:** Figure (1.1) shows the numerical solutions and the exact solutions at time $t=1.0, 2.0, 3.0, 4.0$. From the figure, we can see that the numerical solutions approximate the exact solutions very well with good resolution around shocks.

**Problem 4.2:** Figure (1.2) shows the numerical solutions and the reference solutions at

![Graphs showing numerical and exact solutions](image)

Figure 1.1: Numerical result for Problem 4.1: numerical solution at $N_x=200$ (square symbols) and the exact solution (solid line) at different time $t$. Top left: $t=1.0$; Top right: $t=2.0$; Bottom left: $t=3.0$; Bottom right: $t=4.0$. The small windows in bottom figures are the zoom in version of the corresponding region in the circle.

time $t=1.0, 2.0, 3.0, 4.0$. The reference solutions are the numerical solutions computed at $N_x=2000$. Again, the scheme converges and has good resolution around shock. Figure (1.3) shows the graph of $r$ vs. $f_{HI}$ at $t=1.0, 2.0, 3.0, 4.0$.

**Problem 4.3:** First we compute a reference solution with the mesh sizes $N_x=2000$, $N_r=2000$,
Figure 1.2: Numerical result for Problem 4.2: Numerical solution at Nx=200 (square symbols) and the reference solution at Nx=2000 (solid line) at different time t. Top left: t=1.0; Top right: t=2.0; Bottom left: t=3.0; Bottom right: t=4.0. The small windows in bottom figures are the zoom in version of the corresponding region in the circle.

and the domain length in \( \nu \)-direction as \( \nu_{max} = 10^6 \).

Figure (1.4) shows the numerical solutions at \( Nx=400, N_r=400 \) and reference solutions of \( J(t=1.0, r, \nu) \) at a set of fixed \( r \). From the figure, we can see the convergence of the numerical solution and the dependence of \( J \) on \( r \).

Figure (1.5) are the graphs of \( \nu \text{ vs. } \alpha \) at \( Nx=400, N_r=400 \) at a set of fixed \( r \).

Figure (1.6) shows the numerical solutions at \( Nx=400, N_r=400 \) and reference solutions of \( J(t=1.0, r, \nu) \) at a set of fixed \( \nu \).

Figure (1.7)(1.8)(1.9) are the graphs of \( r \text{ vs. } \alpha \) at \( t=1.0 \) at a set of fixed \( \nu \) with mesh \( Nx=400, \)}
Figure 1.3: The graphs of r vs. $f_{HI}$ for numerical solution at Nx=200 of Problem 4.2 at different time t. Top left: t=1.0; Top right: t=2.0; Bottom left: t=3.0; Bottom right: t=4.0.

$N_v=400$.

Figure (1.10) are the graphs of t vs. $\alpha$ at t=1.0 at a set of fixed r and $\nu$. The $\alpha$ is set to be 1 at $r \geq t$ where J=0. A discontinuity of $\alpha$ occurs around $t = r$, where the discontinuity of J happens.

Figure (1.11) shows the numerical solutions at Nx=400, $N_v=400$ and reference solutions of $J(t=4.0, r, \nu)$ at a set of fixed r. From the figure, we can see the convergence of the numerical solution and the dependence of J on r.

Figure (1.12) are the graphs of $\nu$ vs. $\alpha$ at Nx=400, $N_v=400$ at a set of fixed r.
Figure (1.13) shows the numerical solutions and reference solutions of $J(t=4.0, r, \nu)$ at a set of fixed $\nu$.

Figure (1.14)(1.15)(1.16) are the graphs of $r$ vs. $\alpha$ at $t=4.0$ at a set of fixed $\nu$ with mesh $N_x=400, N_\nu=400$.

Figure (1.17) are the graphs of $t$ vs. $\alpha$ at $t=4.0$ at a set of fixed $r$ and $\nu$. The $\alpha$ is set to be 1 at $r \geq t$ where $J=0$. A discontinuity of $\alpha$ occurs around $t = r$, where the discontinuity of $J$ happens.

By looking into the graphs of numerical solution and reference solution at different fixed $r$ and different fixed $\nu$, we have the convergence of the numerical solution. Hence, we draw the 3-D numerical solution at the mesh: $N_x=400, N_\nu=400$. Figure (1.18) shows the 3-D graph of numerical solutions $J(t, r, \nu)$ as a function of $r, \nu$ at different time $t$. Figure (1.19) shows the graph $r$ vs. $f_{HI}$ at different time $t$. 
Figure 1.4: Numerical result for Problem 4.3: $\nu$(in Log scale).vs.$J$ of numerical solution at $N_x=400$, $N_\nu=400$(square symbols) and the reference solution at $N_x=2000$, $N_\nu=2000$(solid line) at time $t=1.0$, at a set of fixed $r$ with the range of $\nu$ from 1 to 100. Top left: $r=0.2$; Top right: $r=0.5$; Bottom left: $r=0.7$; Bottom right: $r=0.9$. 
Figure 1.5: Numerical result for Problem 4.3: $\nu$ (in Log scale) vs $\alpha$, at $N_x=400$, $N_\nu = 400$ at time $t=1.0$ at a set of fixed $r$. Top left: $r=0.2$; Top right: $r=0.5$; Bottom left: $r=0.7$; Bottom right: $r=0.9$. 
Figure 1.6: Numerical result for Problem 4.3: numerical solution(square symbols), r vs J, at $N_x=400$, $N_\nu = 400$(square symbols) and the reference solution(solid line) at $N_x=2000$, $N_\nu =2000$(solid line) at time $t=1.0$ at a set of fixed $\nu$. Left: $\nu = 1$, right: $\nu = 1001$. 
Figure 1.7: Numerical result for Problem 4.3: r vs. $\alpha$ at $N_x=400$, $N_y=400$ at time $t=1.0$ at a set of fixed $\nu$. Top left: $\nu=1.32$; Top right: $\nu=1.74$; Bottom left: $\nu=2.29$; Bottom right: $\nu=3.02$;
Figure 1.8: Numerical result for Problem 4.3: r vs. $\alpha$ at $N_x=400$, $N_p=400$ at time $t=1.0$ at a set of fixed $\nu$. Top left: $\nu=3.98$; Top right: $\nu=63.10$; Bottom left: $\nu=1000.00$; Bottom right: $\nu=15848.93$;
Figure 1.9: Numerical result for Problem 4.3: $r$ vs. $\alpha$, at $N_x = 400$, $N_\nu = 400$ at time $t=1.0$ at a set of fixed $\nu$. Put all the figures of Fig(1.7)(1.8) in one graph.
Figure 1.10: Numerical result for Problem 4.3: t vs. $\alpha$ at $N_x=400$, $N_r=400$ at a set of fixed $r$ and fixed $\nu$. When $r$ $\geq$ $t$ $(J=0)$, the $\alpha$ is set to be 1. A discontinuity of $\alpha$ occurs around $t=r$, where the discontinuity of $J$ happens. Top left: $r=0.2$, $\nu=1.15$; Top right: $r=0.6$, $\nu=3.98$; Bottom left: $r=0.2$, $\nu=15.85$; Bottom right: $r=0.6$, $\nu=63.10$. 
Figure 1.11: Numerical result for Problem 4.3: numerical solution (square symbols), $\log(\nu)$ vs. $J$, at $N_x=400$, $N_\nu = 400$ (square symbols) and the reference solution (solid line) at $N_x=2000$, $N_\nu=2000$ (solid line) at time $t=4.0$ at a set of fixed $r$. The range of $\nu$ is [1, 100] Top left: $r=1.6$; Top right: $r=2.5$; Bottom left: $r=3.1$; Bottom right: $r=3.7$. 
Figure 1.12: Numerical result for Problem 4.3: \( \nu \) (in Log scale) vs. \( \alpha \) at \( N_x=400, N_\nu=400 \) at time \( t=4.0 \) at a set of fixed \( r \). Top left: \( r=0.8 \); Top right: \( r=2.0 \); Bottom left: \( r=2.8 \); Bottom right: \( r=3.6 \).
Figure 1.13: Numerical result for Problem 4.3: numerical solution(square symbols), r vs. J, at Nx=400, \( N_\nu = 400 \) (square symbols) and the reference solution(solid line) at Nx=2000, \( N_\nu = 2000 \) (solid line) at time \( t=4.0 \) at a set of fixed \( \nu \). Left: \( \nu = 1 \), right: \( \nu = 1001 \). The small windows in left figure is the zoom in version of the corresponding region in the circle.
Figure 1.14: Numerical result for Problem 4.3: r vs. \( \alpha \) at \( N_x = 400, N_r = 400 \) at time \( t = 4.0 \) at a set of fixed \( \nu \). Top left: \( \nu = 1.32 \); Top right: \( \nu = 1.74 \); Bottom left: \( \nu = 2.29 \); Bottom right: \( \nu = 3.02 \);
Figure 1.15: Numerical result for Problem 4.3: $r$ vs. $\alpha$ at $N_x=400$, $N_y = 400$ at time $t=4.0$ at a set of fixed $\nu$. Top left: $\nu=3.98$; Top right: $\nu=63.10$; Bottom left: $\nu=1000.00$; Bottom right: $\nu=15848.93$;
Figure 1.16: Numerical result for Problem 4.3: $r$ vs. $\alpha$, at $N_x=400$, $N_\nu = 400$ at time $t=4.0$ at a set of fixed $\nu$. Put all the figures of Fig(1.14)(1.15) in one graph.
Figure 1.17: Numerical result for Problem 4.3: t vs. α at Nx=400, Nρ = 400 at time t=4.0 at a set of fixed r and fixed ν. When r≥t(J≡0), the α is set to be 1. A discontinuity of α occurs around t = r, where the discontinuity of J happens. Top left: r=0.8, ν=1.15; Top right: r=2.4, ν=3.98; Bottom left: r=2.8, ν=15.85; Bottom right: r=2.4, ν=63.10.
Figure 1.18: 3-D numerical result for Problem 4.3 with mesh $N_x=400$, $N_y=400$, $\nu_{max}=10^6$, at different time $t$. Top left: $t=1.0$; Top right: $t=2.0$; Bottom left: $t=3.0$; Bottom right: $t=4.0$. 
Figure 1.19: Numerical result r vs. $f_{HI}$, with mesh $N_x=400$, $N_y=400$, for Problem 4.3 at different time $t$. Top left: $t=1.0$; Top right: $t=2.0$; Bottom left: $t=3.0$; Bottom right: $t=4.0$. 