High Order Semi-Lagrangian WENO scheme for Vlasov Equations

Jingmei Qiu

Department of Mathematical and Computer Science
Colorado School of Mines
joint work w/ Andrew Christlieb
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Outline

- Vlasov-Poisson system
- Typical numerical approach
- The proposed semi-Lagrangian (SL) WENO method
  - High order accuracy for smooth structure and non-oscillatory resolution for sharp interface
  - Mass conservation
  - Allows for large time step
- Simulation results
The collisionless plasma may be described by the well-known Vlasov-Poisson (VP) system,

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \mathbf{E}(t, \mathbf{x}) \cdot \nabla_v f = 0, \tag{1}
\]

\[
\mathbf{E}(t, \mathbf{x}) = -\nabla_x \phi(t, \mathbf{x}), \quad -\Delta_x \phi(t, \mathbf{x}) = \rho(t, \mathbf{x}). \tag{2}
\]

\(f(t, \mathbf{x}, \mathbf{v})\): the probability of finding a particle with velocity \(\mathbf{v}\) at position \(\mathbf{x}\) at time \(t\).
\(\rho(t, \mathbf{x}) = \int f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} - 1\): charge density
Numerical approach: Lagrangian vs. Eulerian

- Lagrangian: tracking a finite number of macro-particles.
  - e.g., PIC (Particle In Cell)

\[
\frac{dx}{dt} = v, \quad \frac{dv}{dt} = E
\]  

- Eulerian: fixed numerical mesh
  - e.g., finite difference WENO, semi-Lagrangian, finite volume, finite element, spectral method.
Numerical Challenge

- Lagrangian: suffers from the statistical noise ($O(1/\sqrt{N})$).
- Eulerian: suffers the curse of dimensionality for a multi-D (up to 6) problem.

Our approach: A very high order semi-Lagrangian approach with relatively coarse numerical mesh in both space and time.
The SL WENO method

Key Component

- Strang splitting
- Conservative form of semi-Lagrangian (SL) scheme
- WENO reconstruction.
Strang splitting

\[ \frac{\partial f}{\partial t} + v \cdot \nabla_x f + E(t, x) \cdot \nabla_v f = 0, \]

Nonlinear Vlasov equation \( \Rightarrow \) a sequence of linear equations.

- 1-D in \( x \) and 1-D in \( v \):
  \[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0 \], \hspace{1cm} (4)
  \[ \frac{\partial f}{\partial t} + E(t, x) \frac{\partial f}{\partial v} = 0. \] \hspace{1cm} (5)
Strang Splitting for solving the Vlasov equation

1. Evolve equation (4) for $\frac{\Delta t}{2}$,
2. Compute $E$ from the Poisson’s equation,
3. Evolve equation (5) for $\Delta t$,
4. Evolve equation (4) for $\frac{\Delta t}{2}$.
Linear transport equation

\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0 , \]

1. Shifting of solution profile

2. Mass conservation (if the boundary condition is periodic)

\[ d \int f(x, t) dx \frac{dt}{dt} = 0 \]
SL method for linear transport equation

\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0 , \]

*Numerical scheme: evolution solution from \( t^n \) to \( t^{n+1} \)

1. reconstruction: determine accuracy quality of the scheme.
2. shifting: by \( v \Delta t \)

Let \( \xi_0 = v \frac{\Delta t}{\Delta x} \),

\[ f_i^{n+1} = f(x_i, t^{n+1}) = f(x_i - \xi_0 \Delta x, t^n) \]
Third order example

When $\xi_0 \in [0, \frac{1}{2}]$
Third order example

When $\xi_0 \in [0, \frac{1}{2}]$

When $\xi_0 \in \left[-\frac{1}{2}, 0\right]$
When $\xi_0 \in [0, \frac{1}{2}]$

\[
 f_i^{n+1} = f_i^n + \left( -\frac{1}{6} f_{i-2}^n + f_{i-1}^n - \frac{1}{2} f_i^n - \frac{1}{3} f_{i+1}^n \right) \xi_0 \\
+ \left( \frac{1}{2} f_{i-1}^n - f_i^n + \frac{1}{2} f_{i+1}^n \right) \xi_0^2 \\
+ \left( \frac{1}{6} f_{i-2}^n - \frac{1}{2} f_{i-1}^n + \frac{1}{2} f_i^n - \frac{1}{6} f_{i+1}^n \right) \xi_0^3 ,
\]

(6)
Matrix Vector From

\[ f_i^{n+1} = f_i^n + \xi_0(f_{i-2}^n, f_{i-1}^n, f_i^n, f_{i+1}^n) \cdot A_L^3 \cdot (1, \xi_0, \xi_0^2)', \quad (7) \]

with matrix

\[
A_L^3 = \begin{pmatrix}
-\frac{1}{6} & 0 & \frac{1}{6} \\
1 & \frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & -1 & 1 \\
-\frac{1}{3} & \frac{1}{2} & -\frac{1}{6}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-\frac{1}{6} & 0 & \frac{1}{6} \\
\frac{5}{6} & \frac{1}{2} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\
0 & 0 & 0
\end{pmatrix} - \begin{pmatrix}
0 & 0 & 0 \\
-\frac{1}{6} & 0 & \frac{1}{6} \\
\frac{5}{6} & \frac{1}{2} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{2} & \frac{1}{6}
\end{pmatrix}
\]


Conservative Form

\[
f_i^{n+1} = f_i^n - \xi_0 ((f_{i-1}^n, f_i^n, f_{i+1}^n) \cdot C_3^L \cdot (1, \xi_0, \xi_0^2)') - (f_{i-2}^n, f_{i-1}^n, f_i^n) \cdot C_3^L \cdot (1, \xi_0, \xi_0^2)')
\]

with

\[
C_3^L = \begin{pmatrix}
-\frac{1}{6} & 0 & \frac{1}{6} \\
\frac{5}{6} & \frac{1}{2} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{2} & \frac{1}{6}
\end{pmatrix}
\]
\[ f_i^{n+1} = f_i^n - \xi_0 (\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n) \]

**Numerical flux:**

\[
\hat{f}_{i+1/2}^n = (f_{i-1}^n, f_i^n, f_{i+1}^n) \cdot C_3^L \cdot (1, \xi_0, \xi_0^2)' \\
= (f_{i-1}^n, f_i^n, f_{i+1}^n) \cdot C_3^L(:, 1) \\
+ (f_{i-1}^n, f_i^n, f_{i+1}^n) \cdot C_3^L(:, 2) \xi_0 \\
+ (f_{i-1}^n, f_i^n, f_{i+1}^n) \cdot C_3^L(:, 3) \xi_0^2
\]
WENO reconstructions on fluxes

Apply WENO reconstruction on

\[(f_{i-1}^n, f_i^n, f_{i+1}^n) \cdot C^L_3(\cdot, 1)\]

\[= -\frac{1}{6} f_{i-1}^n + \frac{5}{6} f_i^n + \frac{1}{3} f_{i+1}^n\]

\[= \gamma_1 (-\frac{1}{2} f_{i-1}^n + \frac{3}{2} f_i^n) + \gamma_2 (\frac{1}{2} f_i^n + \frac{1}{2} f_{i+1}^n)\]

with linear weights \(\gamma_1 = \frac{1}{3}\) and \(\gamma_2 = \frac{2}{3}\).
WENO reconstructions (cont.)

Idea of WENO:

adjust the linear weighting $\gamma_i$ to a nonlinear weighting $w_i$, such that

- $w_i$ is very close to $\gamma_i$, in the region of smooth structures,
- $w_i$ weights little on a non-smooth sub-stencil.
Nonlinear weights $w_i$

- Smoothness indicator $\beta_i$:

$$\beta_1 = (f^n_{i-1} - f^n_i)^2, \quad \beta_2 = (f^n_i - f^n_{i+1})^2.$$  

- Nonlinear weights $\tilde{w}_i$:

$$\tilde{w}_1 = \gamma_1/((\epsilon + \beta_1)^2), \quad \tilde{w}_2 = \gamma_2/((\epsilon + \beta_2)^2)$$

- Normalized nonlinear weights $w_i$:

$$w_1 = \tilde{w}_1/\tilde{w}_1 + \tilde{w}_2), \quad w_2 = \tilde{w}_2/\tilde{w}_1 + \tilde{w}_2$$
Summary of algorithm

1. Compute the numerical fluxes $\hat{f}_{i+1/2}^n$
   - compute the nonlinear weights $w_1$, $w_2$.
   - compute numerical fluxes

   $$\hat{f}_{i+1/2}^n = w_1\left(-\frac{1}{2}f_{i-1}^n + \frac{3}{2}f_i^n\right) + w_2\left(\frac{1}{2}f_i^n + \frac{1}{2}f_{i+1}^n\right)$$
   $$+ (f_{i-1}^n, f_i^n, f_{i+1}^n) \cdot C_3^L(:, 2)\xi_0$$
   $$+ (f_{i-1}^n, f_i^n, f_{i+1}^n) \cdot C_3^L(:, 3)\xi_0^2$$

2. Update the solution $f_i^{n+1}$

   $$f_i^{n+1} = f_i^n - \xi_0(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n)$$
Mass Conservation

The scheme conserves the total mass, if the periodic boundary conditions are imposed.

**Proof.**

\[
\sum_i f_i^{n+1} = \sum_i \left(f_i^n - \xi_0(\hat{f}_{i+\frac{1}{2}}^n - \hat{f}_{i-\frac{1}{2}}^n)\right)
\]

\[
= \sum_i f_i^n - \xi_0 \sum_i (\hat{f}_{i+\frac{1}{2}}^n - \hat{f}_{i-\frac{1}{2}}^n)
\]

\[
= \sum_i f_i^n.
\]
Comparison

• Compare with the SL method
  • compare with the existing scheme that the integrated mass:
    — very high spatial accuracy in multi-dimensional problem.
  • compare with the existing scheme that the point values:
    — mass conservation.

• Compare with other Eulerian approaches
  • allows for extra large numerical time step integration,
  • commits no error in time for one dimensional problem, but is subject to the splitting error $O(\Delta t^2)$. 
Extensions

- The SL WENO scheme has been extended to 5th, 7th, 9th order in accuracy.
- The scheme is planned to extended to solve the Vlasov-Maxwell system.
- The scheme is planned to couple with the adaptive mesh refinement (AMR) framework to realize spatial adaptivity.
Numerical Simulations
Semi-Lagrangian WENO for Vlasov equation

Jingmei Qiu

Introduction
Proposed Method
Simulation Results
Summary

\[ u_t + u_x = 0 \]

Table: Order of accuracy with \( u(x, t = 0) = \sin(x) \) at \( T = 20 \).
\( CFL = 1.2 \).

<table>
<thead>
<tr>
<th>mesh</th>
<th>3rd order</th>
<th>5th order</th>
<th>9th order</th>
</tr>
</thead>
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<tr>
<td></td>
<td>error</td>
<td>order</td>
<td>error</td>
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<tr>
<td>32</td>
<td>3.27E-2</td>
<td>–</td>
<td>7.31E-5</td>
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<td>64</td>
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<td>1.11E-3</td>
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<td>192</td>
<td>6.81E-4</td>
<td>2.67</td>
<td>9.16E-9</td>
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</table>
$u_t + u_x = 0$

**Figure:** 320 grid points, $CFL = 0.6$, $T = 20$. 
Linear transport: $u_t + u_x + u_y = 0$

Table: Order of accuracy with $u(x, y, t = 0) = \sin(x + y)$ at $T = 20$. $CFL = 1.2$.

<table>
<thead>
<tr>
<th>mesh</th>
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<th>5th order</th>
<th>9th order</th>
</tr>
</thead>
<tbody>
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<td>order</td>
<td>error</td>
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<td>0.96</td>
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<td>2.10</td>
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<tr>
<td>90</td>
<td>2.57E-2</td>
<td>2.08</td>
<td>3.15E-6</td>
</tr>
</tbody>
</table>
Linear transport: $u_t + u_x + u_y = 0$

**Figure:** The numerical mesh is $90 \times 90$. $CFL = 0.6$ at $T = 0.5$. 
Rigid body rotation:

\[ u_t - yu_x + xu_y = 0 \]

**Figure:** The numerical mesh is 90 × 90. \( CFL = 1.2 \) at \( T = 2\pi \).
Consider the VP system with initial condition,

\[ f(x, v, t = 0) = \frac{1}{\sqrt{2\pi}} (1 + \alpha \cos(kx)) \exp(-\frac{v^2}{2}), \quad (8) \]
Weak Landau damping:
\[ \alpha = 0.01, \ k = 2 \]

**Figure:** Time evolution of $L^2$ norm of electric field.
Weak Landau damping (cont.)

Figure: Time evolution of $L^1$ (upper left) and $L^2$ (upper right) norm, discrete kinetic energy (lower left), entropy (lower right).
Strong Landau damping:
\[ \alpha = 0.5, \ k = 2 \]

**Figure:** Time evolution of the \( L^2 \) norm of the electric field using third, fifth, seventh and ninth order reconstruction.
Figure: Phase space plots of strong Landau damping at \( T = 30 \) using third, fifth, seventh and ninth order reconstruction. The numerical mesh is \( 64 \times 128 \).
Two-stream instability

Consider the symmetric two stream instability, the VP system with initial condition

\[ f(x, v, t = 0) = \frac{2}{7\sqrt{2\pi}}(1 + 5v^2)(1 + \alpha((\cos(2kx) + \cos(3kx))/1.2 + \cos(kx))\exp(-\frac{v^2}{2}) , \]

with \( \alpha = 0.01 \), \( k = 0.5 \) and the length of domain in \( x \)-direction \( L = \frac{2\pi}{k} \).
Figure: Two stream instability $T = 53$. The numerical mesh is $64 \times 128$. 
Figure: Third order semi-Lagrangian WENO method. Two stream instability $T = 53$. The numerical mesh is $128 \times 256$ (left) and $256 \times 512$ (right). The computational cost is 4 and 16 times as the first plot in Figure 8.
Figure: Time development of numerical $L^2$ norm (left) and entropy (right) of two stream instability.
Summary

• Equations:
  – the VP system

• Methodology:
  – Strang-split semi-Lagrangian WENO method (3rd, 5th, 7th and 9th order)

• Simulations:
  High order schemes is more efficient than low order ones. It better preserves the energy, entropy of the system.
THANK YOU!