Lecture 4

Section 2.5 The Pinching Theorem  Section 2.6 Two Basic Properties of Continuity

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1 Review

1.1 Continuity

Homework and Quizzes

Homework 1 & 2

• Homework 1 is due September 4th in lab.
• Homework 2 is due September 9th in lab.

Quizzes 1, 2 & 3

• Quizzes 1, 2 and 3 are available on CourseWare!
• Quizzes 1 and 2 are due on this Friday!

Daily Grades

• Daily grades will be given in lecture beginning next Tuesday.
• The daily grades form is posted on the course homepage. You must print out this form and BRING it to class every day.
• Questions will be asked in lecture at random times.
• You will mark your answers on the daily grades form and drop the form in a box at the end of class.

Weekly Written Quizzes in Lab

• Quizzes will be given every week on Thursday in lab beginning the next week.
• The weekly written quizzes form is posted on the course homepage. You must print out this form and BRING it to class every Thursday.
**Definition of Continuity at a Point**

**Definition 1.** Let $f$ be a function defined on an open interval centered at $c$. We say that $f$ is continuous at $c$ if

$$\lim_{x \to c} f(x) = f(c).$$

**Remark**

$f$ is continuous at $c$ if

1. $f$ is defined at $c$,
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$.

**Three Types of “Simple” Discontinuity**

“Removable” Discontinuity

“Jump” Discontinuity
“Infinite” Discontinuity

Example

At which points is $f$ discontinuous? And what type of discontinuity does $f$ have?

Continuity Properties of Elementary Functions

Theorem 2. • The absolute function $f(x) = |x|$ is continuous everywhere.
The square root function \( f(x) = \sqrt{x} \) is continuous at any positive number.

- Polynomials are continuous everywhere.
- Rational Functions are continuous everywhere defined.

Continuity Properties of Sums, Products and Quotients

**Theorem 3.** If \( f \) and \( g \) are continuous at \( c \), then

1. \( f + g \) and \( f - g \) are continuous at \( c \).
2. \( k f \) is continuous at \( c \) for each real \( k \).
3. \( f \cdot g \) is continuous at \( c \).
4. \( f/g \) is continuous at \( c \) provided \( g(c) \neq 0 \).

Continuity Properties of Compositions

**Theorem 4.** If \( g \) is continuous at \( c \) and \( f \) is continuous at \( g(c) \), then the composition \( f \circ g \) is continuous at \( c \).

**Examples 5.**

- \( F(x) = \sqrt{\frac{x^2 + 1}{x - 3}} \) is continuous wherever it is defined, i.e., at any number \( c > 3 \). Note that \( F = f \circ g \) where \( f(x) = \sqrt{x} \) and \( g(x) = \frac{x^2 + 1}{x - 3} \).

One Sided Continuity

**Definition 6.**

- \( f \) is left continuous at \( c \) if \( \lim_{x \to c^-} f(x) = f(c) \).
- \( f \) is right continuous at \( c \) if \( \lim_{x \to c^+} f(x) = f(c) \).

\( f(x) = \sqrt{x} \) is right-continuous at 0.
\[ f(x) = \sqrt{1 - x} \text{ is left-continuous at 1.} \]

**Continuity on Intervals**

**Definition 7.** Let \( I \) be an interval of form: \((a, b), [a, b], (a, b], [a, b), (a, \infty), [a, \infty), (\infty, b), (\infty, b], \text{ or } (\infty, \infty)\). The \( f \) is said to be continuous on \( I \) if for every number \( c \) in \( I \),

- \( f \) is **continuous at** \( c \) if \( c \) is not an endpoint of \( I \),
- \( f \) is **left continuous at** \( c \) if \( c \) is a right-endpoint of \( I \),
- \( f \) is **right continuous at** \( c \) if \( c \) is a left-endpoint of \( I \).

**1.2 Trigonometric Functions**

Trigonometric Functions: Sine and Cosine
Sine and Cosine: Important Identities

Unit Circle

\[ \sin^2 \theta + \cos^2 \theta = 1. \]

Odd/Even Function

\[ \sin(-\theta) = -\sin \theta, \]
\[ \cos(-\theta) = \cos \theta. \]
Addition Formula

\[
\begin{align*}
\sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\
\cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta
\end{align*}
\]

Other Trigonometric Functions: Identities

Continuity

The remaining trigonometric functions

\[
\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}
\]

are all continuous where defined.

Important Identities

\[
\tan^2 x + 1 = \sec^2 x, \quad \cot^2 x + 1 = \csc^2 x.
\]

Other Trigonometric Functions: Graphs
2 Section 2.5 The Pinching Theorem; Trigonometric Limits

2.1 The Pinching Theorem

The Pinching Theorem

**Theorem 8.** Let $p > 0$. Suppose that, for all $x$ such that $0 < |x - c| < p$

$$h(x) \leq f(x) \leq g(x).$$

If

$$\lim_{x \to c} h(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = L.$$ 

then

$$\lim_{x \to c} f(x) = L.$$

**The Pinching Theorem: Continuity of Sine and Cosine**

**Theorem 9.**

$$\lim_{x \to 0} \sin x = 0, \quad \lim_{x \to 0} \cos x = 1,$$

$$\lim_{x \to c} \sin x = \sin c, \quad \lim_{x \to c} \cos x = \cos c.$$

$$0 < |\sin x| < |x|.$$
The Pinching Theorem: Trigonometric Limits

Theorem 10.

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0.
\]

Proof.
Use Geometric argument to get

\[\cos x < \frac{\sin x}{x} < 1,\]

then apply the pinching theorem.

3 Section 2.6 Two Basic Properties of Continuous Functions

3.1 The Intermediate-Value Theorem

The Intermediate-Value Theorem
Theorem 11. If $f$ is continuous on $[a, b]$ and $K$ is any number between $f(a)$ and $f(b)$, then there is at least one number $c$ in the interval $(a, b)$ such that $f(c) = K$.

The Intermediate-Value Theorem: Roots of Equation

Theorem 12. If $f$ is continuous on $[a, b]$ and

$$f(a) < 0 < f(b), \quad \text{or} \quad f(b) < 0 < f(a),$$

then the equation $f(x) = 0$ has at least a root in $(a, b)$.

3.2 The Extreme-Value Theorem

The Extreme-Value Theorem

Theorem 13. A function $f$ continuous on a bounded closed $[a, b]$ takes on both a maximum value $M$ and a minimum value $m$. 
The Extreme-Value Theorem: Bounded Closed Intervals

Theorem 14. Continuous functions map bounded closed intervals $[a, b]$ onto bounded closed intervals $[m, M]$. 

\[ f: [a, b] \rightarrow [m, M] \]