Lecture 19

Section 5.6 Indefinite Integrals

Section 5.7 The \( u \)-Substitution

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What is today?

a. Monday
b. Wednesday
c. Friday
d. None of these
Definite Integral

Fundamental Theorem of Integral Calculus

In general,

\[ \int_{a}^{b} f(x) \, dx = [F(x)]_{a}^{b}. \]

where \( F(x) \) is an antiderivative of \( f(x) \), i.e., \( F'(x) = f(x) \).

<table>
<thead>
<tr>
<th>Function</th>
<th>Antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^r )</td>
<td>( \frac{x^{r+1}}{r+1} ) (( r ) a rational number ( \neq -1 ))</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( -\cos x )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( \sin x )</td>
</tr>
<tr>
<td>( \sec^2 x )</td>
<td>( \tan x )</td>
</tr>
<tr>
<td>( \sec x \tan x )</td>
<td>( \sec x )</td>
</tr>
<tr>
<td>( \csc^2 x )</td>
<td>( -\cot x )</td>
</tr>
<tr>
<td>( \csc x \cot x )</td>
<td>( -\csc x )</td>
</tr>
</tbody>
</table>
Give the value of $\int_{-1}^{2} (2x^2 - 3x - 5) \, dx$.

a. $-12$

b. $-13$

c. $\frac{-27}{2}$

d. $\frac{-29}{2}$

e. None of these
Definite Integral as Signed Area

\[ \int_{a}^{b} f(x) \, dx \]

\[ = \int_{a}^{c} f(x) \, dx + \int_{c}^{d} f(x) \, dx + \int_{d}^{e} f(x) \, dx + \int_{e}^{b} f(x) \, dx \]

\[ = \text{Area of } \Omega_1 + (\text{Area of } \Omega_2) + \text{Area of } \Omega_3 + (\text{Area of } \Omega_4) \]

\[ = (\text{Area of } \Omega_1 + \text{Area of } \Omega_3) - (\text{Area of } \Omega_2 + \text{Area of } \Omega_4) \]

\[ = \text{Area above the } x\text{-axis} - \text{Area below the } x\text{-axis}. \]
The graph of $y = f(x)$ is shown below, where $\Omega_1$ has area 2, $\Omega_2$ has area 3, $\Omega_3$ has area 8, and $\Omega_4$ has area 4. Give $\int_a^b f(x) \, dx$.

a. 1  
b. 3  
c. 12  
d. 14  
e. None of these
The graph of $y = f(x)$ is shown below, where $\Omega_1$ has area 2, $\Omega_2$ has area 3, $\Omega_3$ has area 8, and $\Omega_4$ has area 4. Give $\int_a^e f(x) \, dx$.

a. 1

b. 3

c. 12

d. 14

e. None of these
The graph of $y = f(x)$ is shown below, where $\Omega_1$ has area 2, $\Omega_2$ has area 3, $\Omega_3$ has area 8, and $\Omega_4$ has area 4. Give the area bounded by the graph of $f$ and the $x$-axis for $d \leq x \leq b$.

a. 1  

b. 3  

c. 12  

d. 14  

e. None of these
The Indefinite Integral of $f$

$$\int f(x) \, dx = \text{The general anti-derivative of } f.$$ 

Example

$$\int x^2 \, dx = \frac{1}{3}x^3 + C$$

where $C$ is an arbitrary constant.

The indefinite integral $f$ is a family of antiderivatives of $f$. 
The Indefinite Integral of $f$

In general,

$$\int f(x) \, dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and $C$ is an arbitrary constant.

Example

$$\int \sqrt{x} \, dx = \frac{2}{3} x^{3/2} + C.$$  

The indefinite integral $f$ is a family of antiderivatives of $f$. 
The Indefinite Integral of $f$

In general,

$$\int f(x) \, dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and $C$ is an arbitrary constant.

\[
\begin{align*}
\int x^r \, dx &= \frac{x^{r+1}}{r+1} + C \quad (r \text{ rational, } r \neq -1) \\
\int \sin x \, dx &= -\cos x + C \\
\int \sec^2 x \, dx &= \tan x + C \\
\int \csc^2 x \, dx &= -\cot x + C
\end{align*}
\]

\[
\begin{align*}
\int \cos x \, dx &= \sin x + C \\
\int \sec x \tan x \, dx &= \sec x + C \\
\int \csc x \cot x \, dx &= -\csc x + C
\end{align*}
\]
Further Properties of Integral Calculus

Theorem

1. \[ \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx \]

2. \[ \int \alpha f(x) \, dx = \alpha \int f(x) \, dx, \quad \alpha \text{ a constant} \]

In general,

\[ \int (\alpha f(x) + \beta g(x)) \, dx = \alpha \int f(x) \, dx + \beta \int g(x) \, dx \]

where \( \alpha \) and \( \beta \) are constants.
Example

Calculate \( \int (\frac{5}{2}x^{3/2} - 2 \csc^2 x) \, dx \)
Example

Find $f$ given that $f'(x) = x^3 + 2$ and $f(0) = 1$.

Note that, since $f'$ is the derivative of $f$, $f$ is an antiderivative for $f'$:

$$f(x) = \int f'(x) \, dx$$

The constant of integration $C$ can be evaluated using the fact that $f(0) = 1$. 
Note: Most differentiation involves the chain rule, so we should expect that most antidifferentiation will involve

undoing the chain rule

(the $u$-Substitution)

If $F$ is an antiderivative for $f$, then

$$\frac{d}{dx}[F(u(x))] = F'(u(x)) \cdot u'(x) = f(u(x)) \cdot u'(x)$$

$$\int f(u(x)) \cdot u'(x) \, dx = \int f(u) \, du = F(u) + C = F(u(x)) + C.$$
## Undoing the Chain Rule

<table>
<thead>
<tr>
<th>$F(x)$</th>
<th>$F'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{n+1} [u(x)]^{n+1}$</td>
<td>$[u(x)]^n u'(x)$</td>
</tr>
<tr>
<td>$\sin[u(x)]$</td>
<td>$\cos[u(x)] u'(x)$</td>
</tr>
<tr>
<td>$\cos[u(x)]$</td>
<td>$- \sin[u(x)] u'(x)$</td>
</tr>
<tr>
<td>$\tan[u(x)]$</td>
<td>$\sec^2[u(x)] u'(x)$</td>
</tr>
<tr>
<td>$\cot[u(x)]$</td>
<td>$- \csc^2[u(x)] u'(x)$</td>
</tr>
<tr>
<td>$\sec[u(x)]$</td>
<td>$\sec[u(x)] \tan[u(x)] u'(x)$</td>
</tr>
<tr>
<td>$\csc[u(x)]$</td>
<td>$- \csc[u(x)] \cot[u(x)] u'(x)$</td>
</tr>
</tbody>
</table>
### The $u$-Substitution

| Original Integral | $u = g(x)$,
$du = g'(x)dx$ | New Integral |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int [g(x)]'g'(x) , dx$</td>
<td>$\int u' , du = \frac{u^{r+1}}{r+1} + C = \frac{[g(x)]^{r+1}}{r+1} + C \quad (r \neq -1)$</td>
<td></td>
</tr>
<tr>
<td>$\int \sin [g(x)] , g'(x) , dx$</td>
<td>$\int \sin u , du = -\cos u + C = -\cos [g(x)] + C$</td>
<td></td>
</tr>
<tr>
<td>$\int \cos [g(x)] , g'(x) , dx$</td>
<td>$\int \cos u , du = \sin u + C = \sin [g(x)] + C$</td>
<td></td>
</tr>
<tr>
<td>$\int \sec^2 [g(x)] , g'(x) , dx$</td>
<td>$\int \sec^2 u , du = \tan u + C = \tan [g(x)] + C$</td>
<td></td>
</tr>
<tr>
<td>$\int \sec [g(x)] \tan [g(x)]g'(x) , dx$</td>
<td>$\int \sec u , \tan u , du = \sec u + C = \sec [g(x)] + C$</td>
<td></td>
</tr>
<tr>
<td>$\int \csc^2 [g(x)] , g'(x) , dx$</td>
<td>$\int \csc^2 u , du = -\cot u + C = -\cot [g(x)] + C$</td>
<td></td>
</tr>
<tr>
<td>$\int \csc [g(x)] \cot [g(x)] , g'(x) , dx$</td>
<td>$\int \csc u , \cot u , du = -\csc u + C = -\csc [g(x)] + C$</td>
<td></td>
</tr>
</tbody>
</table>
Examples

Calculate

1. \[ \int (x^2 - 1)^4 2x \, dx \]
2. \[ \int 3x^2 \cos(x^3 + 2) \, dx \]
3. \[ \int \sin x \cos x \, dx \]
4. \[ \int \frac{dx}{(3 + 5x)^2} \]
Examples

Calculate

5. \[ \int x^2 \sqrt{4 + x^3} \, dx \]

6. \[ \int 2x^3 \sec^2(x^4 + 1) \, dx \]

7. \[ \int \sec^3 x \tan x \, dx \]

8. \[ \int x(x - 3)^5 \, dx \]
Substitution in Definite Integrals

The Change of Variables Formula

\[ \int_{a}^{b} f(u(x))u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du. \]

We change the limits of integration to reflect the substitution.
Evaluate

1. \[ \int_{0}^{2} (x^2 - 1)(x^3 - 3x + 2)^3 \, dx \]

2. \[ \int_{0}^{1/2} \cos^3 \pi x \sin \pi x \, dx \]

3. \[ \int_{0}^{\sqrt{3}} x^5 \sqrt{x^2 + 1} \, dx \]