Test 2 Review

Jiwen He

Department of Mathematics, University of Houston

`jiwenhe@math.uh.edu`
`math.uh.edu/~jiwenhe/Math1431`

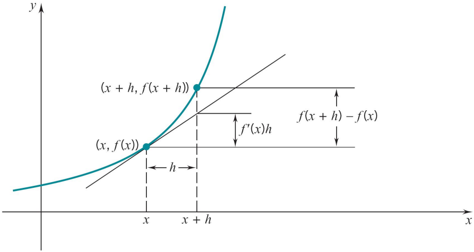
Online Quizzes

All current and previous quizzes in Math 1431 are now opened until November 4th.

- Review for Test 2 by Prof. Morgan.
- Thursday 8:00 - 10:00pm in 100 SEC

Good Sources of Practice Problems

- Examples from class.
- The basic homework problems.
- The basic online quiz problems.



- increment: $\Delta f = f(x+h) - f(x)$
- differential: $df = f'(x)h$

$$\Delta f \approx df$$

in the sense that $\dfrac{\Delta f - df}{h}$ tends to 0 as $h \to 0$.

Quiz 1

Use differentials to estimate $\sqrt{26}$, by using your knowledge of $\sqrt{25}$.

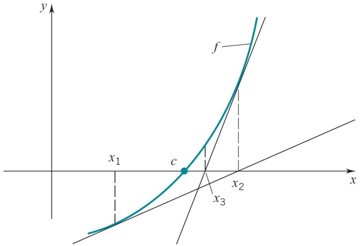
a. 5.15

b. 5.05

c. 5.1

d. 5.2

e. None of these



Newton Method

Let the number c be a solution (root) of an equation $f(x) = 0$. The Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \cdots,$$

generates a sequence of approximations $x_1, x_2, \cdots, x_n, \cdots$ that will "converge" to the root c

Quiz 2

Use 1 iteration of Newton's method to estimate $\sqrt{26}$, starting from a guess of 5, by noting that $\sqrt{26}$ is a root of $x^2 - 26 = 0$.

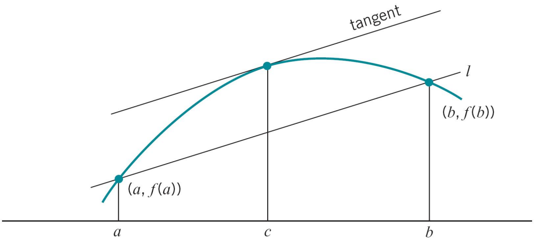
a. 5.15

b. 5.05

c. 5.1

d. 5.2

e. None of these



Theorem

If f is differentiable on the open interval (a, b) and continuous on the closed interval $[a, b]$, then there is at least one number c in (a, b) for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or equivalently

$$f(b) - f(a) = f'(c)(b - a).$$

Quiz 3

Give the number of values in $(0, 2\pi)$ where the MVThm is satisfied.

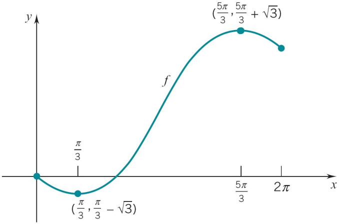
a. 0

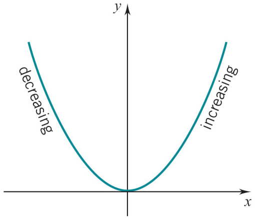
b. 1

c. 2

d. 3

e. None of these

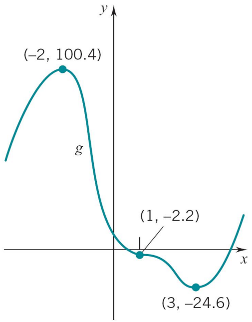




Theorem

- *A function f is increasing on an interval I if*
  - *f is continuous and*
  - *$f'(x) > 0$ at all but finitely many values in I.*
- *A function f is decreasing on an interval I if*
  - *f is continuous and*
  - *$f'(x) < 0$ at all but finitely many values in I.*

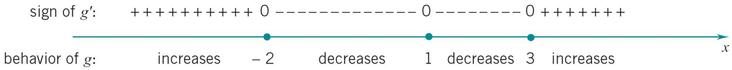
Example



$$f(x) = \frac{4}{5}x^5 - 3x^4 - 4x^3 + 22x^2 - 24x + 6,$$

$$f'(x) = 4(x+2)(x-1)^2(x-3)$$

- f is continuous everywhere.
- f is increasing on $(-\infty, -2]$, decreasing on $[-2, 3]$, and increasing on $[3, \infty)$.



Quiz 4

Assume the domain of f is all real numbers. The graph of $f'(x)$ is shown below. Give the number of intervals of increase of f.

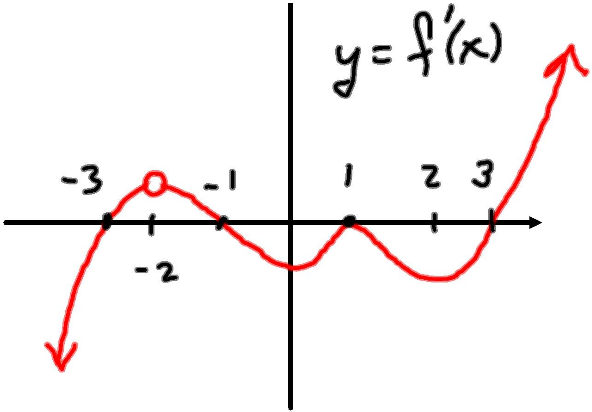
a. 1

b. 2

c. 3

d. 4

e. None of these



Quiz 5

Assume the domain of f is all real numbers. The graph of $f'(x)$ is shown below. Give the number of intervals of decrease of f.

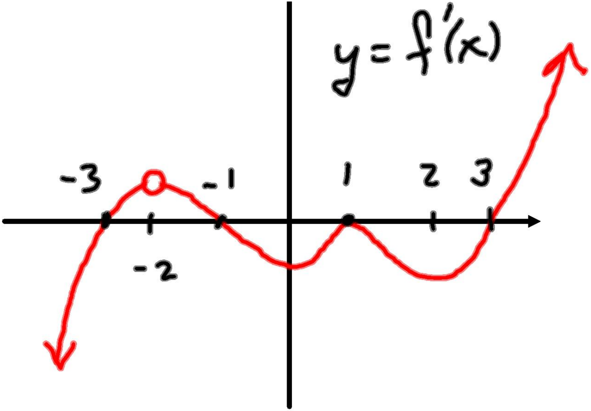
a. 1

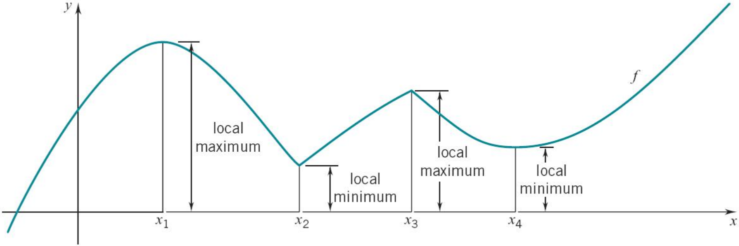
b. 2

c. 3

d. 4

e. None of these





THEOREM 4.3.2

If f takes on a local maximum or minimum at c, then either

$$f'(c) = 0 \qquad \text{or} \qquad f'(c) \text{ does not exist.}$$

DEFINITION 4.3.3 CRITICAL NUMBER

The numbers c in the domain of a function f for which either

$$f'(c) = 0 \qquad \text{or} \qquad f'(c) \text{ does not exist,}$$

are called the *critical numbers of* f. †

Assume the domain of f is all real numbers. The graph of $f'(x)$ is shown below. Give the number of critical values of f.

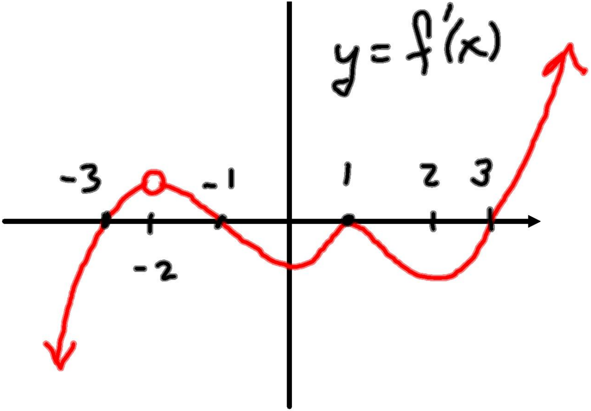
a. 2

b. 3

c. 4

d. 5

e. None of these



Section 4.3 First Derivative Test

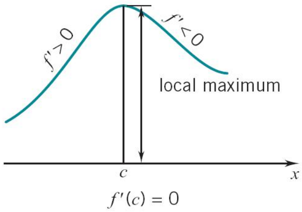


Figure 4.3.7

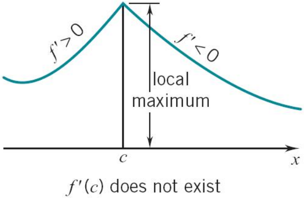


Figure 4.3.8

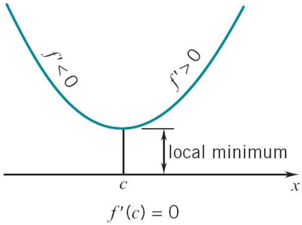


Figure 4.3.9

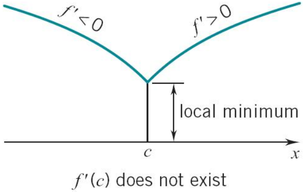


Figure 4.3.10

THEOREM 4.3.5 THE SECOND-DERIVATIVE TEST

Suppose that $f'(c) = 0$ and $f''(c)$ exists.

(i) If $f''(c) > 0$, then $f(c)$ is a local minimum.

(ii) If $f''(c) < 0$, then $f(c)$ is a local maximum.

Assume the domain of f is all real numbers. The graph of $f'(x)$ is shown below. Give the number of local minima of f.

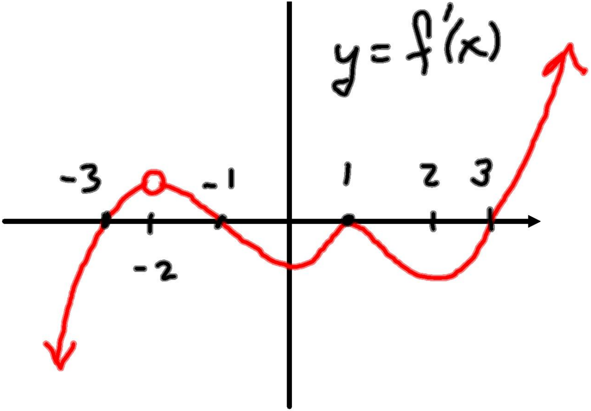
a. 1

b. 2

c. 3

d. 4

e. None of these



Quiz 8

Assume the domain of f is all real numbers. The graph of $f'(x)$ is shown below. Give the number of local maxima of f.

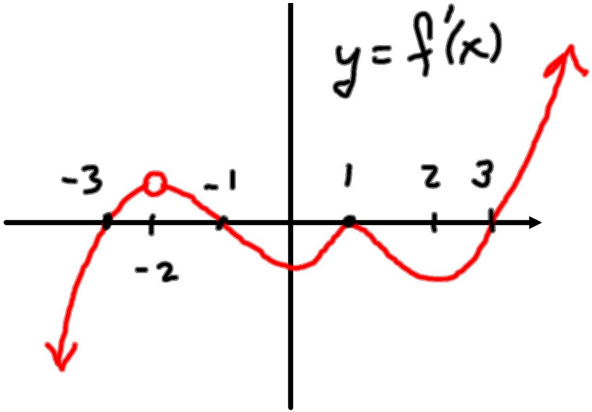
a. 1

b. 2

c. 3

d. 4

e. None of these



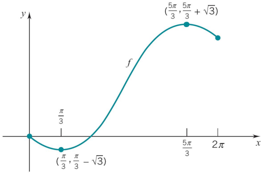
Section 4.4 Absolute Max/Min of f on $[a, b]$

Step 1. Find the critical numbers $c_1, c_2, \cdots$ of f in the open interval (a, b).

Step 2. Calculate $f(a)$, $f(c_1)$, $f(c_2)$, $\cdots$, $f(b)$.

Step 3. The largest of the numbers found in step 2 is the absolute maximum value of f and the smallest is the absolute minimum.

Example: Abosolute Max/Min of f on $[a, b]$

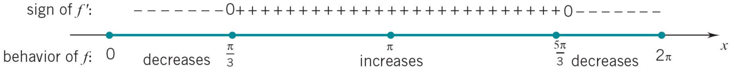


$$f(x) = x - 2\sin x, \quad 0 \le x \le 2\pi,$$

$$f'(x) = 1 - 2\cos x, \quad 0 \le x \le 2\pi.$$

$$f'(x) = 0 \text{ at } x = \pi/3, 5\pi/3.$$

- f is continuous on $[0, 2\pi]$.
- f is decreasing on $[0, \pi/3]$, increasing on $[\pi/3, 5\pi/3]$, and decreasing on $[5\pi/3, 2\pi]$.



Section 4.4 Absolute Max/Min of f on $[a, \infty)$ or $(-\infty, b]$

Step 1. Find the critical numbers — the numbers c at which $f'(c) = 0$ or $f'(c)$ does not exist.

Step 2. Test each endpoint of the domain by examining the sign of the first derivative nearby.

Step 3. Test each critical number c by examining the sign of the first derivative on both sides of c (first-derivative test) or by checking the sign of the second derivative at c itself (second derivative test).

Step 4. If the domain of f is unbounded, determine the behavior of f as $x \to \infty$ or as $x \to -\infty$.

Step 5. Determine whether any of the endpoint extremes and local extremes are absolute extremes.

Assume the domain of f is all real numbers. The graph of $f'(x)$ is shown below. Give the number of absolute minima of f.

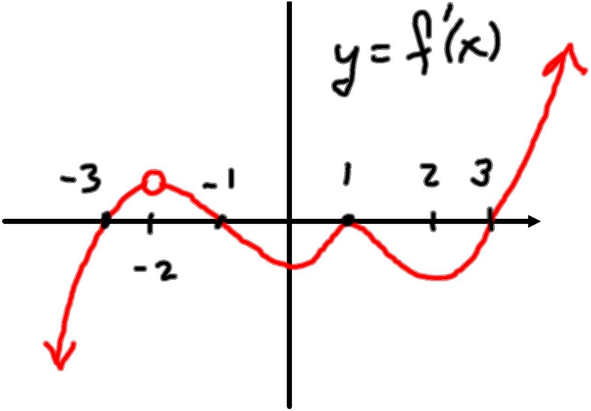
a. 1

b. 2

c. 3

d. 4

e. None of these



Assume the domain of f is all real numbers. The graph of $f'(x)$ is shown below. Give the number of absolute maxima of f.

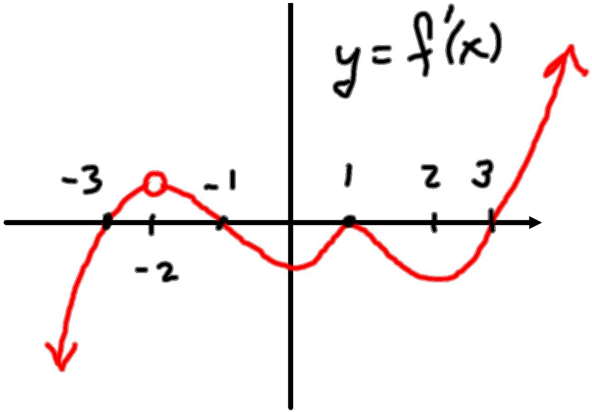
a. 1

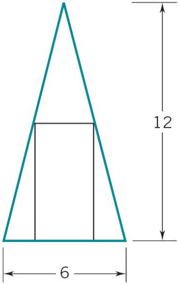
b. 2

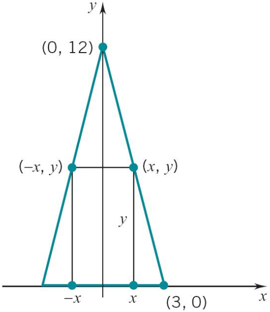
c. 3

d. 4

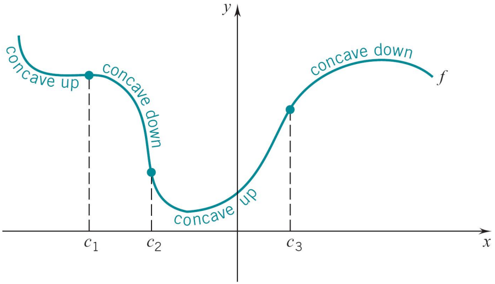
e. None of these







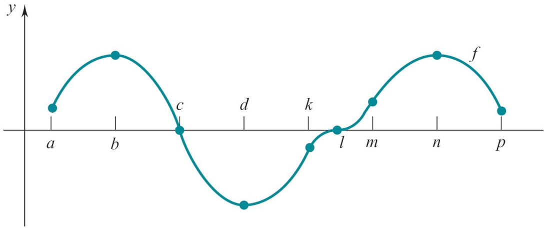
Example 1 An isosceles triangle has a base of 6 units and a height of 12 units. Find the maximum possible area of a rectangle that can be placed inside the triangle with one side resting on the base of the triangle. What are the dimensions of the rectangle(s) of maximum area?



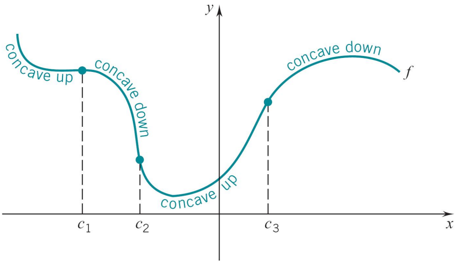
Definition

- The graph of f is concave up on I if f' increases on I.
- The graph of f is concave down on I if f' decreases on I.
- Ponts that join arcs of opposite concavity are points of inflection.

Example



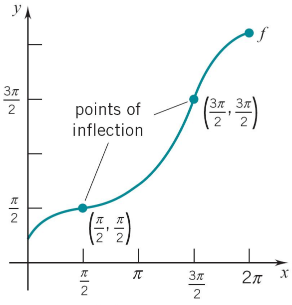
- Determine the intervals on which f increases and the intervals on which f decreases.
- Determine the intervals on which the graph of f is concave up and the intervals on which the graph of f is concave down.
- Give the x-coordinates of the points of inflection.



Theorem

- *If $f''(x) > 0$ for all x in I, then f' increases on I, and the graph of f is concave up.*
- *If $f''(x) < 0$ for all x in I, then f' decreases on I, and the graph of f is concave down.*
- *If the point $(c, f(c))$ is a point of inflection, then either $f''(c) = 0$ or $f'(c)$ does not exist.*

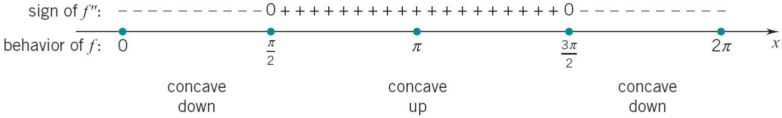
Example



$f(x) = x + \cos x$

- Determine concavity and find the points of inflection of the graph of $f(x) = x + \cos x,\ x \in [0, 2\pi]$.

$$f'(x) = 1 - \sin x,\ f''(x) = -\cos x.$$

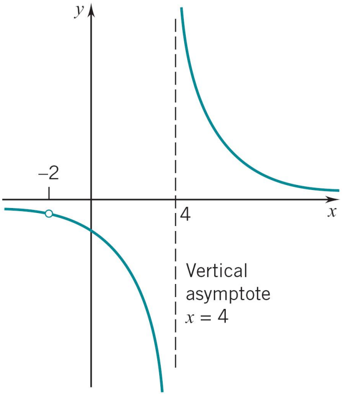


Typically, to locate the vertical asymptotes for a function f,

- find the values $x = c$ at which f is discontinuous
- and determine the behavior of f as x approaches c.

The vertical line $x = c$ is a vertical asymptote for f if any one of the following conditions holds

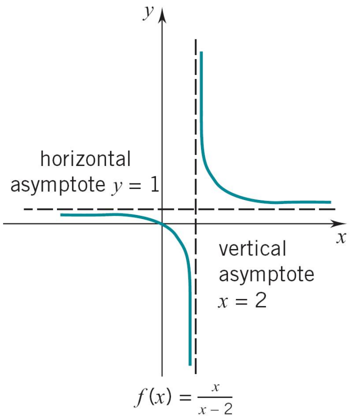
- $f(x) \to \infty$ or $-\infty$ as $x \to c^+$;
- $f(x) \to \infty$ or $-\infty$ as $x \to c^-$;
- $f(x) \to \infty$ or $-\infty$ as $x \to c$.



The line $x = 4$ is a vertical asymptote for

$$f(x) = \frac{3x+6}{x^2-2x-8} = \frac{3(x+2)}{(x+2)(x-4)}.$$

Section 4.7 Aymptotes: Rational Function $f(x) = \frac{x}{x-2}$



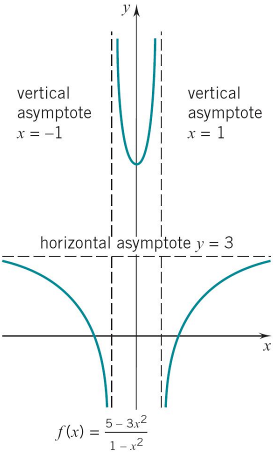
- The line $x = 2$ is a vertical asymptote.
- The line $y = 1$ is a horizontal asymptote.

Let

$$R(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_k x^k + \cdots + b_1 x + b_0}$$

be a rational function. Then

- if $n < k$,
$$R(x) \to 0 \quad \text{as} \quad x \to \pm\infty;$$
- if $n = k$,
$$R(x) \to \frac{a_n}{b_n} \quad \text{as} \quad x \to \pm\infty;$$
- if $n > k$,
$$R(x) \to \pm\infty \quad \text{as} \quad x \to \pm\infty.$$



- The lines $x = \pm 1$ are vertical asymptotes.
- The line $y = 3$ is a horizontal asymptote.

Sketch the graph of f

- Step 1: Domain of f
- Step 2: Intercepts
- Step 3: Symmetry and Periodicity
- Step 4: First Derivative f'
- Step 5: Second Derivative f''
- Step 6: Preliminary sketch
- Step 7: Sketch the graph

Problem 1

Use differentials to estimate $\sqrt{102}$.

Problem 2

Use differentials to estimate $\sin 46^o$.

Problem 3

A metal sphere with a radius of 10 cm is to be covered with a 0.02 cm coating of silver. Use differentials to estimate how much silver will be required.

Problem 4

Use one iteration of Newton's method to approximate $\sqrt{48}$ from a guess of 7. Hint: $\sqrt{48}$ is a root of $x^2 - 48 = 0$.

Problem 5

Use one iteration of Newton's method to approximate a root of $2x^3 + 4x^2 - 8x + 3 = 0$.

Problem 6

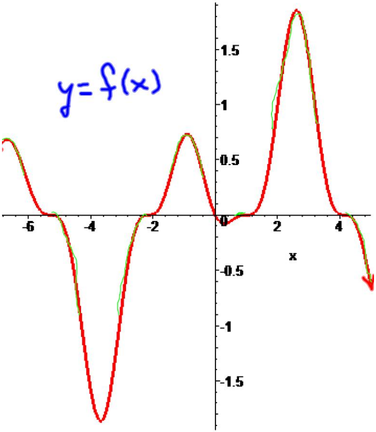
Verify the conclusion of the mean value theorem for $f(x) = x^3 - 4x^2 + x + 6$ on the interval $[-1, 2]$.

Problem 7

A car is stationary at a toll booth. Twenty minutes later, at a point 20 miles down the road, the car is clocked at 60 mph. Use the mean value theorem to explain why the car must have exceeded the 60 mph speed limit at some time after leaving the toll booth, but before the car was clocked at 60 mph.

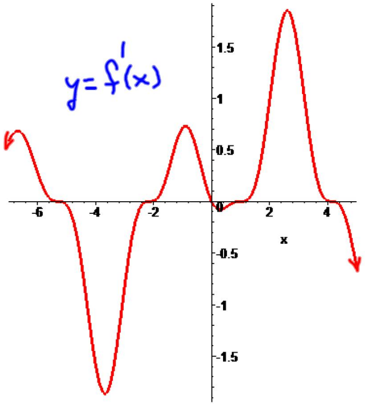
Problem 8

The graph of $f(x)$ is shown below. Give the intervals of increase, decrease, concave up and concave down. Also find and classify any critical numbers and list any values where the function has inflection.



Problem 9

The graph of $f'(x)$ is shown below. Give the intervals of increase, decrease, concave up and concave down. Also find and classify any critical numbers and list any values where the function has inflection.



Problem 10

Classify the critical numbers of $f(x) = x^2 + x + 3|x| - 3$.

Problem 11

The function $f(x) = x^2 \cos(x)$ has a critical number at $x = 0$. Use the second derivative test to classify this critical number.

Problem 12

Give the maximun and minimum values for $f(x) = -x^3 + 6x^2 + 15x - 2$ on the interval $[-2, 1]$.

Problem 13

What are the dimensions of the base of the rectangular box of greatest volume that can be constructed from 100 square inches of cardboard if the base is to be twice as long as it is wide? Assume that the box has top.

Problem 14

A rectangular playground is to be fenced off and divied into two parts by a fence parallel to one side of the playground. Six hundred feet of fencing is used. Find the dimensions of the playground that will enclose the greatest total area.

Problem 15

Give the horizontal and vertical asymptotes of $f(x) = \dfrac{2x^2 - x - 1}{x^2 + x - 2}$.

Problem 16

Graph $f(x) = \frac{3}{5}x^{5/3} - 3x^{2/3}$. Be sure to plot any critical points, points of inflection and the y intercept, etc. Also, indicate the intervals of increase, decrease, concave up and concave down.

Problem 16 (cont.)

Graph of $f(x) = \frac{3}{5}x^{5/3} - 3x^{2/3}$

Step 1: Domain of f

(i) Determine the domain of f;

(ii) Identify endpoints;

(iii) Find the vertical asymptotes;

(iv) Determine the behavior of f as $x \to \pm\infty$;

(v) Find the horizontal asymptotes.

Problem 16 (cont.)

Sketch the graph of $f(x) = \frac{3}{5}x^{5/3} - 3x^{2/3}$

Step 2: Intercepts

(i) Determine the y-intercept of the graph:
- The y-intercept is the value of $f(0)$;

(ii) Determine the x-intercepts of the graph:
- The x-intercepts are the solutions of the equation $f(x) = 0$.

Problem 16 (cont.)

Sketch the graph of $f(x) = \dfrac{3}{5}x^{5/3} - 3x^{2/3}$

Step 3: Symmetry and Periodicity

(i) Symmetry:

(a) If f is an even function, i.e., $f(-x) = f(x)$, then the graph is symmetric w.r.t. the y-axis;

(b) If f is an odd function, i.e., $f(-x) = -f(x)$, then the graph is symmetric w.r.t. the origin.

(ii) Periodicity:

- If f is periodic with period p, then the graph replicates itself on intervals of length p.

Problem 16 (cont.)

Sketch the graph of $f(x) = \frac{3}{5}x^{5/3} - 3x^{2/3}$

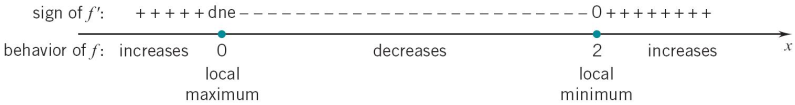
Step 4: First Derivative f'

(i) Calculate f';

(ii) Determine the critical numbers of f;

(iii) Examine the sign of f' to determine the intervals on which f increases and the intervals on which f decreases;

(iv) Determine vertical tangents and cusps.



Problem 16 (cont.)

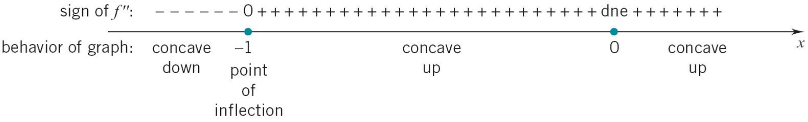
Sketch the graph of $f(x) = \dfrac{3}{5}x^{5/3} - 3x^{2/3}$

Step 5: Second Derivative f''

(i) Calculate f'';

(ii) Examine the sign of f'' to determine the intervals on which the graph is concave up and the intervals on which the graph is concave down;

(iii) Determine the points of inflection.



Problem 16 (cont.)

Sketch the graph of $f(x) = \frac{3}{5}x^{5/3} - 3x^{2/3}$

Step 6: Preliminary sketch

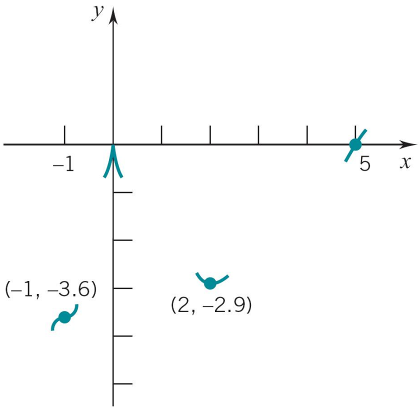
Plot the points of interest:

(i) intercept points,

(ii) extreme points

- local extreme points,
- endpoint extreme points,
- absolute extreme points,

(iii) and points of inflection.



Problem 16 (cont.)

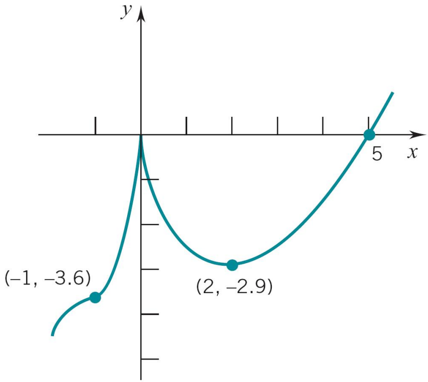
Sketch the graph of $f(x) = \frac{3}{5}x^{5/3} - 3x^{2/3}$

Step 7: Sketch the graph

(i) Neither symmetry and nor periodicity;

(ii) Connect the points of the preliminary sketch;

(iii) Make sure the curve "rises", "falls", and "bends" in the proper way.



Problem 17

Graph $f(x) = \sin 2x - 2\sin x$. Be sure to plot any critical points, points of inflection and the y intercept, etc. Also, indicate the intervals of increase, decrease, concave up and concave down.

Problem 17

Sketch the graph of $f(x) = \sin 2x - 2\sin x$.

Step 1: Domain of f

(i) Determine the domain of f;

(ii) Identify endpoints;

(iii) Find the vertical asymptotes;

(iv) Determine the behavior of f as $x \to \pm\infty$;

(v) Find the horizontal asymptotes.

Sketch the graph of $f(x) = \sin 2x - 2\sin x$

Step 2: Intercepts

(i) Determine the y-intercept of the graph:
- The y-intercept is the value of $f(0)$;

(ii) Determine the x-intercepts of the graph:
- The x-intercepts are the solutions of the equation $f(x) = 0$.

Problem 17 (cont.)

Sketch the graph of $f(x) = \sin 2x - 2\sin x$

Step 3: Symmetry and Periodicity

(i) Symmetry:

(a) If f is an even function, i.e., $f(-x) = f(x)$, then the graph is symmetric w.r.t. the y-axis;

(b) If f is an odd function, i.e., $f(-x) = -f(x)$, then the graph is symmetric w.r.t. the origin.

(ii) Periodicity:

- If f is periodic with period p, then the graph replicates itself on intervals of length p.

Problem 17 (cont.)

Sketch the graph of $f(x) = \sin 2x - 2\sin x$

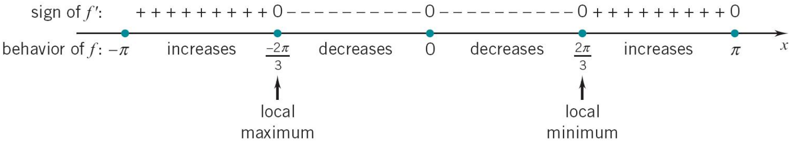
Step 4: First Derivative f'

(i) Calculate f';

(ii) Determine the critical numbers of f;

(iii) Examine the sign of f' to determine the intervals on which f increases and the intervals on which f decreases;

(iv) Determine vertical tangents and cusps.



Problem 17 (cont.)

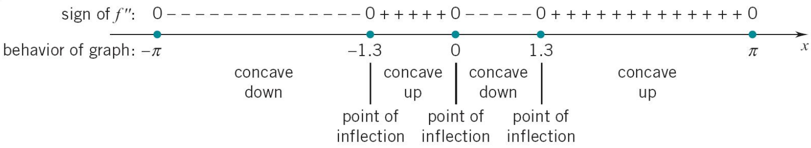
Sketch the graph of $f(x) = \sin 2x - 2\sin x$

Step 5: Second Derivative f''

(i) Calculate f'';

(ii) Examine the sign of f'' to determine the intervals on which the graph is concave up and the intervals on which the graph is concave down;

(iii) Determine the points of inflection.



Sketch the graph of $f(x) = \sin 2x - 2\sin x$

Step 6: Preliminary sketch

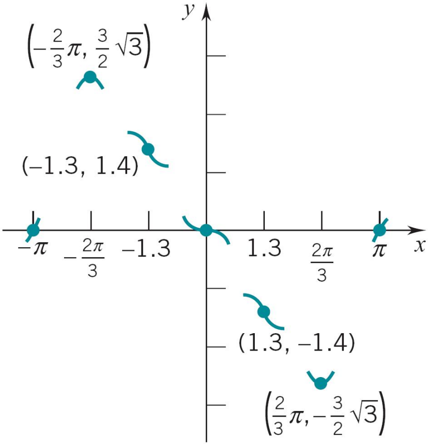
Plot the points of interest:

(i) intercept points,

(ii) extreme points

- local extreme points,
- endpoint extreme points,
- absolute extreme points,

(iii) and points of inflection.



Problem 17 (cont.)

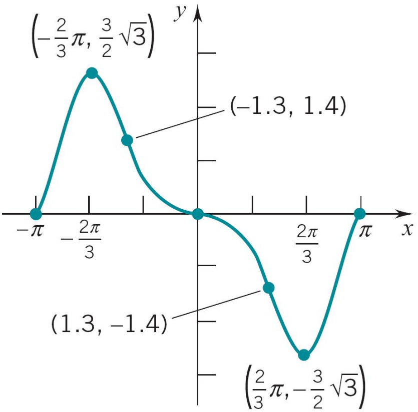
Sketch the graph of $f(x) = \sin 2x - 2\sin x$

Step 7: Sketch the graph

(i) Symmetry: sketch the graph on the interval $[-\pi, \pi]$;

(ii) Connect the points of the preliminary sketch;

(iii) Make sure the curve "rises", "falls", and "bends" in the proper way;



Sketch the graph of $f(x) = \sin 2x - 2\sin x$

Step 7: Sketch the graph

(iv) Obtain the complete graph by replicating itself on intervals of length 2π.